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# Geometry of interactions in complex bodies

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**Abstract**

We analyze geometrical structures necessary to represent bulk and surface interactions of standard and substructural nature in complex bodies. Our attention is mainly focused on the influence of diffuse interfaces on sharp discontinuity surfaces. In analyzing this phenomenon, we prove the covariance of surface balances of standard and substructural interactions.


*Keywords:* Multifield theories; Complex materials; Discontinuity surfaces; Lie symmetries

## 1. Introduction

Bodies are called *complex* when their material substructure (i.e. the texture from nano-level to meso-level) has a prominent influence on their gross behavior and there is a not negligible occurrence of interactions due to substructural changes. Examples are liquid crystals, elastomers, ferroelectric and microcracked bodies, spin glasses. Above all, soft condensed matter displays complex behavior. Applications in nanotechnology, smart structures and various fields of technology open basic theoretical and experimental problems that

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challenge, in a certain sense, even some aspects of the foundational concepts of standard continuum mechanics.

Basically, the standard paradigm of Cauchy's format of continuum mechanics, prescribing that the material element is a sort of indistinct sphere that we collapse in a point in space seems to be not sufficient to account for the articulated substructural nature of a complex body.

In fact, for complex bodies the material element is rather a 'system' and one needs the introduction of an appropriate *morphological descriptor* $\boldsymbol{\nu}$ of such a system (order parameter), at least at a coarse grained level, that describes the essential geometrical features of substructural shapes.

Physical circumstances of disparate nature suggest many possible choices of $\boldsymbol{\nu}$, each one characterizing special models. Moreover, the selection of morphological descriptors is strongly related with the representation of substructural interactions arising within each material element and between neighboring material elements as a consequence of substructural changes. Interactions are represented in fact by objects conjugated in the sense of power with the rates of the quantities describing the geometry of the body and its changes. In this sense, since placement and order parameter fields are involved, the description of complex bodies adopted here is called *multifield*. It has basic differences with standard internal variable models. In a multifield approach, morphological descriptors enter directly the geometrical representation of the body and its kinematics; true interactions are associated with their rates and balanced. Information about the material substructure are then introduced already at the level of geometrical description of the body. On the contrary, in standard internal variable models, the geometrical description of the body is of Cauchy's type: the material element is morphologically equivalent to an indistinct sphere described just by its place in space. There, internal variables come into play to describe just the removal from thermodynamical equilibrium (a detailed treatment of these classes of models can be found in [34]).

Here, by following the general unified framework of multifield theories proposed by Capriz in 1989 [4] (see also [3]) and then developed further on in its abstract structure [32,23,8,9], we do not specify the nature of $\boldsymbol{\nu}$. We require only that $\boldsymbol{\nu}$ be an element of a finite-dimensional differentiable paracompact manifold $\mathcal{M}$ without boundary to cover a class as large as possible of special theories. Our attention is then focused on the general setting which contains as special cases prominent theories interpreting problems typical of condensed matter physics. In a certain sense our work deals with a model of models. However, in our general point of view, we face the basic difficulty that $\mathcal{M}$ does not coincide with a linear space in general. Moreover, we cannot consider a priori $\mathcal{M}$ embedded in a linear space. In fact, since the possible embedding of a finite dimensional manifold in a linear space is not unique, its potential choice is of constitutive nature.

We focus our attention on conservative processes. For them, the relevant appropriate Hamiltonian formalism has been developed in [9] as a natural evolution of the Hamiltonian formalism in classical non-linear elasticity (the one discussed in [27]). We start from the results in [9] and analyze some of the rather subtle geometrical questions induced by the abstract nature of $\mathcal{M}$.

Our essential point of view is as in what follows: “Geometry and mechanics associated with maps between manifolds are a general framework for condensed matter physics and are also a tool to construct new models of unusual and perhaps unexpected phenomena” [25].

Here, we focus the attention on the interaction between diffuse interfaces and additional sharp discontinuity surfaces. From one hand, in fact, in complex bodies there is a frequent occurrence of branching of substructures between domain walls and/or homophase gradient effects. The presence of the gradient of the morphological descriptor $\boldsymbol{\nu}$ in the list of entries of the Lagrangian density allows us to account for these effects ‘smearing’ them as due to diffused interfaces. However, from the other hand, additional macroscopic surfaces of discontinuity may also occur and evolve. They are due to defects such as crack, shock or acceleration waves; their evolution is influenced by the presence of diffused interfaces due to substructural arrangements. As an example one may consider a polarized ferroelectric material in which external loads induce a shock wave: the shock front encounters walls of polarized domains and interact with them. Also, such domain walls influence the propagation of surface defects like cracks, as experiments point out.

In general, when sharp discontinuity surfaces are endowed with own energy, they are referred to as *structured*; on the contrary they are *unstructured*. We analyze here both cases paying attention to the nature of interface balances of standard and substructural actions that involve the jump of bulk stresses and, in the structured case, surface stresses. Really, interface balances involving standard surface stresses have been obtained in [17] while surface substructural measures of interactions have been introduced in [22,23] and the relevant balance equations derived there (see also [8]). However, there is no proof of their covariance. Such a proof is provided here in Theorem 2 and is the main result of this paper. It implies a non-standard notion of observer, which is, for us, not only the representation of the ambient space and the time scale, but also the representation of the manifold of substructural shapes (see also [25]).

The technique of the proof is based on the validity of an integral balance of energetic nature. In the case of unstructured discontinuity surfaces, such a balance is just the integral version of Noether theorem in the bulk and arises naturally from invariance properties of the Lagrangian density. In the structured case, such a balance is augmented by the energetic contribution of the discontinuity surface given in terms of a superficial flux of energy the physical nature of which is discussed in Section 4.

In Section 2 we discuss the natural way to represent the morphology of complex bodies (a way necessary when the prominence of substructural interactions renders not efficient standard homogenization techniques). We describe the use of morphological descriptors to represent the geometry of substructural shapes and discuss up to a certain extent the nature of the space of maps assigning to each material element its morphological descriptor. In Section 3 we start to construct mechanics by assigning a Lagrangian density in which substructural gradient effects are taken into account and recall from [9] the version of Noether theorem appropriate to multifield descriptions of complex bodies. Finally, Section 4 contains the main result, i.e. the proof of the covariance of interfacial balances of standard and substructural interactions.

## 2. Morphology of complex bodies

We consider a body occupying a regular region $\mathcal{B}_0$ of the three-dimensional Euclidean space $\mathcal{E}^3$ (with affine translation space Vec).[1] The current morphology of the body is described by two sufficiently smooth mappings:

$$\mathcal{B}_0 \ni \mathbf{X} \overset{\tilde{\mathbf{x}}}{\longmapsto} \mathbf{x} = \tilde{\mathbf{x}}(\mathbf{X}) \in \mathcal{E}^3, \qquad \mathcal{B}_0 \ni \mathbf{X} \overset{\tilde{\boldsymbol{\nu}}}{\longmapsto} \boldsymbol{\nu} = \tilde{\boldsymbol{\nu}}(\mathbf{X}) \in \mathcal{M}. \tag{1}$$

1. $\tilde{\mathbf{x}}$ shows the current *placement* of a material element at $\mathbf{X}$ in $\mathcal{B}_0$, is injective and orientation preserving, and $\mathcal{B} = \mathbf{x}(\mathcal{B}_0)$ is also regular as $\mathcal{B}_0$.
2. $\tilde{\boldsymbol{\nu}}$ is the map assigning to each material element a coarse grained *morphological descriptor* $\tilde{\boldsymbol{\nu}}(\mathbf{X})$ of its substructure (order parameter), chosen as an element of a differentiable paracompact manifold $\mathcal{M}$ (generally without boundary).

Two natural tangent maps arise, namely $T\tilde{\mathbf{x}} : T\mathcal{B}_0 \to T\mathcal{B}$ and $T\tilde{\boldsymbol{\nu}} : T\mathcal{B}_0 \to T\mathcal{M}$. The pairs $(\mathbf{x}, \nabla\mathbf{x})$, $(\boldsymbol{\nu}, \nabla\boldsymbol{\nu})$ are the peculiar elements of $T\tilde{\mathbf{x}}$ and $T\tilde{\boldsymbol{\nu}}$ respectively. Since $T\mathcal{B}_0$ is a trivial bundle and a connection is natural over it, we can separate in invariant way $\mathbf{x}$ from $\nabla\mathbf{x}$ which is commonly indicated with $\mathbf{F}$. The condition that $\tilde{\mathbf{x}}$ be orientation preserving implies that, at each $\mathbf{X} \in \mathcal{B}_0$, $\mathbf{F} \in \mathrm{Hom}(T_{\mathbf{X}}\mathcal{B}_0, T_{\tilde{\mathbf{x}}(\mathbf{X})}\mathcal{B})$ *has positive determinant*. Since $\mathcal{M}$ is a priori not trivial, the pair $(\boldsymbol{\nu}, \nabla\boldsymbol{\nu})$ cannot be separated in invariant way, unless there is a parallelism over $\mathcal{M}$. In principle, one may define in abstract way a parallelism over $\mathcal{M}$ but, since the pair $(\boldsymbol{\nu}, \nabla\boldsymbol{\nu})$ enters constitutive issues, one should have a physically significant parallelism. In other words, when we act separating in invariant way $\nabla\boldsymbol{\nu}$ from $\boldsymbol{\nu}$ we should presume to have at least one physically significant parallelism over $\mathcal{M}$ even when circumstances may allow us the use of invariance requirements with respect to the choice of the connection. In fact, the presence of the pair $(\boldsymbol{\nu}, \nabla\boldsymbol{\nu})$ or of arbitrarily one element of it (namely $\boldsymbol{\nu}$ or $\nabla\boldsymbol{\nu}$) in the list of entries of the energy changes the representation of interactions.

As remarked in the introduction, in dealing with the geometrical (morphological) description of complex bodies, we relax one of the axioms of the mechanics of simple materials. In our picture, in fact, the material element is not morphologically equivalent to a 'monad', a simple material particle in the sense of Cauchy (see [29,30]), identified only by its place in space. Instead, we consider the body as a collection of *subsystems* of the *same* nature (the material substructure) and the order parameter at a given point represents the characteristic features of the morphology of the subsystem there.

Sometimes the material substructure is a perfectly identifiable Lagrangian system as in the case of nematic liquid crystals [14,20,11] (in which stick molecules can be separated from the melt), sometimes it does not as in granular gases [6] and microcracked bodies [26]. In granular gases, e.g., a material element collects a family of sparse granules with peculiar velocities, so that the order parameter could be an element of a suitable Grassmanian of the tangent bundle of some finite-dimensional manifold, while for microcracked bodies each microcrack can be considered either as a sharp planar defect not interpenetrated by

[1] With the adjective 'regular' we refer to an open (bounded) subset of $\mathcal{E}^3$ coinciding with the interior of its closure, with a surface-like boundary where the outward unit normal $\mathbf{n}$ is well-defined everywhere to within a finite number of corners and edges. The treatment of infinite bodies requires only some minor technical adjustments in the results of the present paper. In any case, some remarks about them are presented throughout the paper.

interatomic bonds or as an elliptic void, so it does not exist per se but it is just determined by the surrounding matter.

We leave undetermined the specific nature of $\boldsymbol{\nu}$ to cover a class as large as possible of special cases, following in this way the unifying point of view of Capriz [4]. Our primary strategy is to work with minimal requirements for $\mathcal{M}$ and to add geometrical structure to it only when necessary.

In this way we avoid the path leading to a collection of special models with slightly different formal aspects but with the same intrinsic nature.

With respect to elasticity of simple bodies (see [27], Chapter 4), here the main source of difficulties is the circumstance that $\mathcal{M}$ is a non-trivial manifold; in particular *it does not coincide* with a linear space in most cases of prominent interest.

The space $\mathcal{C}$ of pairs of maps $(\tilde{\mathbf{x}}, \tilde{\boldsymbol{\nu}})$, a product space of the type $\mathcal{C}_{\mathbf{x}} \times \mathcal{C}_{\boldsymbol{\nu}}$, with $\tilde{\mathbf{x}}$ pertaining to $\mathcal{C}_{\mathbf{x}}$ and $\tilde{\boldsymbol{\nu}}$ to $\mathcal{C}_{\boldsymbol{\nu}}$, has a non-trivial structure which depends on the geometrical properties of $\mathcal{M}$. Basically, we imagine that $\mathcal{C}_{\mathbf{x}} \subseteq W^{1,p}(\mathcal{B}_0, \mathrm{Vec})$ for some $p \geq 1$ and $\mathcal{C}_{\boldsymbol{\nu}} = PC^1(\mathcal{B}_0, \mathcal{M})$, i.e. we require that the order parameter map be at least continuous and piecewise continuously differentiable over $\mathcal{B}_0$, while $\tilde{\mathbf{x}}$ be an element of the Sobolev space $W^{1,p}(\mathcal{B}_0, \mathrm{Vec})$, even if we may basically require that $\tilde{\mathbf{x}}$ be continuous and piecewise continuously differentiable too.

Specific examples showing the possible intricate nature of $\mathcal{C}$, in particular of $\mathcal{C}_{\boldsymbol{\nu}}$, can be discussed at length.

We analyze below the case in which physical circumstances justify a Riemannian structure for $\mathcal{M}$. However, the developments in subsequent sections do not require strictly such a structure that will be called upon only when necessary.

In any case, even when $\mathcal{M}$ is Riemannian, we do not assign a priori any prevalent rôle to Levi–Civita connection.

### 2.1. The case in which $\mathcal{M}$ has a Riemannian structure: aspects of the nature of $\mathcal{C}$

We assume just in this section that $\mathcal{M}$ be Riemannian with metric $\mathbf{g}_{\mathcal{M}}$ and associated Levi–Civita connection.

Notice that *we do not require here the embedding of $\mathcal{M}$ in some linear space*. When, in fact, we embed $\mathcal{M}$ in a linear space by using, let say, Nash's isometric embedding to preserve at least the quadratic part of the kinetic energy (if it exists) pertaining to the substructure, the embedding is not unique. So, the choice of it becomes matter of modeling. Moreover, the embedding is also not strictly necessary to build up the structures needed for the basic aspects of mechanics. For this reason, with the aim to eliminate overstructures as much as possible, we consider $\mathcal{M}$ in its abstract setting, not embedded a priori in a linear space.

We denote with $\langle \cdot, \cdot \rangle_{T\mathcal{M}}$ the scalar product over $T_{\boldsymbol{\nu}}\mathcal{M}$ associated with $\mathbf{g}_{\mathcal{M}}$. For any $C^1$-curve $[0, s^*] \ni s \longmapsto \boldsymbol{\nu}(s) \in \mathcal{M}$, we have a vector field $\boldsymbol{v}(s) = (\mathrm{d}\boldsymbol{\nu}/\mathrm{d}s)(s)$, and for any other vector field $A(s)$ over $\mathcal{M}$ one defines its derivative along $s \longmapsto \boldsymbol{\nu}(s)$ writing $(\mathrm{D}A/\mathrm{d}s) = \nabla_{\boldsymbol{v}(s)}A$, and call $A$ *autoparallel along* $s \longmapsto \boldsymbol{\nu}(s)$ when $(\mathrm{D}A/\mathrm{d}s) = 0$ for any $s \in [0, s^*]$. As usual, $s \longmapsto \boldsymbol{\nu}(s)$ is a *geodesic* when $s \longmapsto \boldsymbol{v}(s)$ is autoparallel along it.

The Riemannian structure assures the existence of a natural distance $d_{\mathcal{M}} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ over $\mathcal{M}$. In fact, by indicating with $\lambda$ any arbitrary piecewise $C^1$ curve $\lambda : [0, s^*] \to \mathcal{M}$

such that $\lambda(0) = \boldsymbol{v}_1$ and $\lambda(s^*) = \boldsymbol{v}_2$, as usual we put

$$d_{\mathcal{M}}(\boldsymbol{v}_1, \boldsymbol{v}_2) = \inf\{l(\lambda)\}, \tag{2}$$

varying $\lambda$ between $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, with $l(\lambda)$ the length

$$l(\lambda) = \int_0^{s^*} \left\langle \frac{\mathrm{d}\lambda}{\mathrm{d}s}(s), \frac{\mathrm{d}\lambda}{\mathrm{d}s}(s) \right\rangle_{T\mathcal{M}}^{(1/2)} \mathrm{d}s \tag{3}$$

(see, e.g. [1] Chapter 2, [33] Chapter 4 ). Locally, $d_{\mathcal{M}}$ is calculated over geodesics and may be unbounded. In this case, we select a new metric $\tilde{d}_{\mathcal{M}} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ over $\mathcal{M}$ defined by

$$\tilde{d}_{\mathcal{M}} = \frac{d_{\mathcal{M}}}{1 + d_{\mathcal{M}}}, \tag{4}$$

which is equivalent to $d_{\mathcal{M}}$ (it leads to the same topology), is bounded, is of class $C^\infty(\mathcal{M})$, if $d_{\mathcal{M}}$ is complete on $\mathcal{M}$, then $\tilde{d}_{\mathcal{M}}$ is complete too. Moreover, the following statements are equivalent: (i) $d_{\mathcal{M}}$ is complete, (ii) closed bounded sets with respect to $d_{\mathcal{M}}$ are compact, (iii) geodesics for $d_{\mathcal{M}}$ can be continued as to be defined on the whole real line ([19], p. 56). If $\mathcal{M}$ is complete, the distance $d_{\mathcal{M}}$ can be computed over geodesics not only locally, but also globally.

**Remark 1.** When $\mathcal{M}$ is compact, $d_{\mathcal{M}} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ is bounded, then it is not necessary to substitute it with $\tilde{d}_{\mathcal{M}}$.

By making use of $\tilde{d}_{\mathcal{M}}$ we may define three natural distances over $\mathcal{C}_{\boldsymbol{v}}$.

1. The first metric distance, indicated with $d^{(\mathrm{i})}$, may be defined by
$$d^{(\mathrm{i})}(\tilde{\boldsymbol{v}}_1, \tilde{\boldsymbol{v}}_2) = \int_{\mathcal{B}_0} \tilde{d}_{\mathcal{M}}(\boldsymbol{v}_1, \boldsymbol{v}_2)\, \mathrm{d(vol)}, \tag{5}$$
where (i) stands for 'integral'.
2. Let $\{K_n\}$ be an exhaustion of $\mathcal{B}_0$ , i.e. a compact cover of $\mathcal{B}_0$ such that $K_n \subset \mathring{K}_{n+1}$ for any $n$. A second distance, indicated with $d^{(\mathrm{c})}$, may be defined by
$$d^{(\mathrm{c})}(\tilde{\boldsymbol{v}}_1, \tilde{\boldsymbol{v}}_2) = \sum_{n\in\mathbb{N}} \frac{1}{2^{-n}} \max_{\mathbf{X}\in K_n} \tilde{d}_{\mathcal{M}}(\boldsymbol{v}_1, \boldsymbol{v}_2), \tag{6}$$
where ($c$) stands for 'compact'.
3. More simply, a third natural distance, indicated with $d^{(\mathrm{s})}$, may be defined by
$$d^{(\mathrm{s})}(\tilde{\boldsymbol{v}}_1, \tilde{\boldsymbol{v}}_2) = \sup_{\mathbf{X}\in\mathcal{B}_0} \tilde{d}_{\mathcal{M}}(\boldsymbol{v}_1, \boldsymbol{v}_2), \tag{7}$$
where ($s$) stands for 'supremum'.

**Remark 2.** There is a peculiar physical difference between the metric $d^{(\mathrm{c})}$ and the other two: the exhaustion of $\mathcal{B}_0$ by compact sets, used to define $d^{(\mathrm{c})}$, implies that the values of the order parameter map over boundary points of $\mathcal{B}_0$ may contribute slightly to the distance in $\mathcal{C}_{\boldsymbol{v}}$, on the contrary of the other two metrics. In a certain sense, $d^{(\mathrm{c})}$ seems to be preferable

when there are uncertainties in the physical meaning of boundary data about $\boldsymbol{\nu}$, a problem (the one of boundary data) that appears very subtle for some material substructures like microcracks.

**Remark 3.** Notice that $d^{(c)}$ and $d^{(s)}$ are complete over $C^0(\mathcal{B}_0, \mathcal{M})$ if $\tilde{d}_{\mathcal{M}}$ is complete. In particular, compactness of $\mathcal{M}$ is sufficient for the validity of this statement.

**Remark 4.** The space of continuous maps between $\mathcal{B}_0$ and $\mathcal{M}$, namely $C(\mathcal{B}_0, \mathcal{M})$, may be in general *not complete* with respect to the metric $d^{(i)}$. Examples can be constructed when $\mathcal{M}$ is not compact and when it is compact. Just to mention a simple case, let us consider a body coinciding with a cube made of a porous solid containing superfluid helium for half of the volume of voids ($\mathcal{M}$ coincides with $S^1 \subset \mathbb{C}$, thus it is compact). In this case we may construct a sequence $\tilde{\nu}_k \in C(\mathcal{B}_0, \mathcal{M})$ that converges in the metric $d^{(i)}$ to a discontinuous function describing the percolation of Helium up to saturate half of the cube.

**Remark 5.** The metric $d^{(i)}$ in (5) may furnish unreasonable results for unbounded bodies. In fact, if we consider an infinite beam made of a two-phase material ($\mathcal{M} = [0, 1]$), we may found two distributions of the two phases (say $\tilde{\nu}_1$ and $\tilde{\nu}_1$ with $\nu$ the volume fraction of one phase) differing just by mirror symmetry such that $d^{(i)}(\tilde{\boldsymbol{\nu}}_1, \tilde{\boldsymbol{\nu}}_2) = +\infty$. In the same case $d^{(c)}$ and $d^{(s)}$ (see (6) and (7)) furnish bounded results.

To control the behavior of the derivatives of $\boldsymbol{\nu}$, basically of $\nabla\tilde{\boldsymbol{\nu}}$ with $\nabla\boldsymbol{\nu} \equiv \nabla\tilde{\boldsymbol{\nu}}(\mathbf{X}) \in \mathrm{Hom}(T_{\mathbf{X}}\mathcal{B}_0, T_{\boldsymbol{\nu}}\mathcal{M})$, we find problematic to act directly on $\mathrm{Hom}(T\mathcal{B}_0, T\mathcal{M})$. In fact, when we select $\tilde{\boldsymbol{\nu}}_1$ and $\tilde{\boldsymbol{\nu}}_2$, at a given $\mathbf{X}$, we get in general $\boldsymbol{\nu}_1 \neq \boldsymbol{\nu}_2$ so that we get $\nabla\boldsymbol{\nu}_1 \in \mathrm{Hom}(T_{\mathbf{X}}\mathcal{B}_0, T_{\boldsymbol{\nu}_1}\mathcal{M})$ and $\nabla\boldsymbol{\nu}_2 \in \mathrm{Hom}(T_{\mathbf{X}}\mathcal{B}_0, T_{\boldsymbol{\nu}_2}\mathcal{M})$. Though $\mathbf{g}_{\mathcal{M}}$ induces natural metric structures over $T^*\mathcal{M}$ and tensor product bundles of $T\mathcal{M}$ and $T^*\mathcal{M}$, it does not give a natural way of comparing $\nabla\boldsymbol{\nu}_1$ and $\nabla\boldsymbol{\nu}_2$ because $\nabla\boldsymbol{\nu}$ is in certain sense like the two-point tensor $\mathbf{F}$. To compare them it is then necessary to transport $\nabla\boldsymbol{\nu}_2$ over $\boldsymbol{\nu}_1$ through a connection or $\nabla\boldsymbol{\nu}_1$ over $\boldsymbol{\nu}_2$. Since we do not assign any prevalent rôle to Levi–Civita connection, even in the case in which $\mathcal{M}$ is complete and we may connect $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ with a geodesic, we face the circumstance that the transport is in general not isometric, so that the comparison (defined in some way) of the two derivatives at $\boldsymbol{\nu}_1$ may lead to a different result of the same comparison at $\boldsymbol{\nu}_2$. Moreover, the transport could be unbounded or the bound could not be uniform over the curve connecting $\boldsymbol{\nu}_1$ with $\boldsymbol{\nu}_2$. The possible choice to consider admissible only values over $\mathcal{M}$ that can be connected by curves assuring a uniformly bounded transport could reduce too much the generality of $\mathcal{M}$.

By considering previous remarks, we consider first the adjoint of $\nabla\tilde{\boldsymbol{\nu}}$, indicated with $\nabla\tilde{\boldsymbol{\nu}}^*$ and such that $\nabla\boldsymbol{\nu}^* \equiv \nabla\tilde{\boldsymbol{\nu}}^*(\mathbf{X}) \in \mathrm{Hom}(T^*_{\boldsymbol{\nu}_1}\mathcal{M}, T^*_{\mathbf{X}}\mathcal{B}_0)$, then we define

$$\bar{d}_{\mathcal{M}}(\nabla\boldsymbol{\nu}_1, \nabla\boldsymbol{\nu}_2) = \|\nabla\boldsymbol{\nu}_1^*\nabla\boldsymbol{\nu}_1 - \nabla\boldsymbol{\nu}_2^*\nabla\boldsymbol{\nu}_2\|, \tag{8}$$

where $\|\cdot\|$ is the usual norm in $\mathbb{R}^3 \otimes \mathbb{R}^3$, i.e. the usual norm of $3 \times 3$ matrices such that, for any $\mathbf{A} \in \mathbb{R}^3 \otimes \mathbb{R}^3$, one has $\|\mathbf{A}\| = \sqrt{\mathrm{tr}(\mathbf{A}^{\mathrm{T}}\mathbf{A})}$ and $\nabla\tilde{\boldsymbol{\nu}}^*\nabla\tilde{\boldsymbol{\nu}}$ is the pull-back in $\mathcal{B}_0$ of the metric $\mathbf{g}_{\mathcal{M}}$ in $\mathcal{M}$, in coordinates $(\nabla\tilde{\boldsymbol{\nu}}^*\nabla\tilde{\boldsymbol{\nu}})_{AB} = \nabla\tilde{\boldsymbol{\nu}}^{*\alpha}_A(\mathbf{g}_{\mathcal{M}})_{\alpha\beta}\nabla\tilde{\boldsymbol{\nu}}^{\beta}_B$.

**Remark 6.** $\bar{d}_{\mathcal{M}}(\nabla\boldsymbol{\nu}_1, \nabla\boldsymbol{\nu}_2)$ compares values of the metric $\mathbf{g}_{\mathcal{M}}$ at two different points (namely $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$) of $\mathcal{M}$. The scalar product $\nabla\boldsymbol{\nu}_1^*\nabla\boldsymbol{\nu}_1 \cdot (\mathrm{d}\mathbf{X} \otimes \mathrm{d}\mathbf{X})$ is the 'length' of $\mathrm{d}\boldsymbol{\nu}$ at $\boldsymbol{\nu}_1$.

With $\bar{d}_{\mathcal{M}}$ as above, we define the counterparts of the distances (5)–(7).

4. The first distance, indicated with $\overline{d}^{(\mathrm{i})}$, is defined by

$$\overline{d}^{(\mathrm{i})}(\nabla\tilde{\boldsymbol{\nu}}_1, \nabla\tilde{\boldsymbol{\nu}}_2) = \int_{\mathcal{B}_0} \overline{d}_{\mathcal{M}}(\nabla\boldsymbol{\nu}_1, \nabla\boldsymbol{\nu}_2)\,\mathrm{d}(\mathrm{vol}). \tag{9}$$

5. Let $\{K_n\}$ be an exhaustion of $\mathcal{B}_0$, i.e. a compact cover of $\mathcal{B}_0$ such that $K_n \subset \mathring{K}_{n+1}$ for any $n$. The second distance, indicated with $\overline{d}^{(\mathrm{c})}$, is defined by

$$\overline{d}^{(\mathrm{c})}(\nabla\tilde{\boldsymbol{\nu}}_1, \nabla\tilde{\boldsymbol{\nu}}_2) = \sum_{n\in\mathbb{N}} \frac{1}{2^{-n}} \max_{\mathbf{X}\in K_n} \overline{d}_{\mathcal{M}}(\nabla\boldsymbol{\nu}_1, \nabla\boldsymbol{\nu}_2). \tag{10}$$

6. More simply, a third natural distance, indicated with $d^{(\mathrm{s})}$, can be defined by

$$\overline{d}^{(\mathrm{s})}(\nabla\tilde{\boldsymbol{\nu}}_1, \nabla\tilde{\boldsymbol{\nu}}_2) = \sup_{\mathbf{X}\in\mathcal{B}_0} \overline{d}_{\mathcal{M}}(\nabla\boldsymbol{\nu}_1, \nabla\boldsymbol{\nu}_2), \tag{11}$$

when $\sup_{\mathbf{X}\in\mathcal{B}_0} \overline{d}_{\mathcal{M}}$ remains bounded over $\mathcal{B}_0$.

**Remark 7.** We do not introduce any 'normalization' of the distance like the one in (4) because (9)–(11) are calculated over bundles whose fibers are linear spaces so that $\overline{d}_{\mathcal{M}}(\cdot,\cdot)$ displays possible properties of homogeneity (natural over fiber spaces) while a normalization like (4) would not.

Appropriate topologies on $\mathcal{C}_{\boldsymbol{\nu}}$ may then be induced by the distances

$$\tilde{d}^{(\mathrm{i})}(\tilde{\boldsymbol{\nu}}_1, \tilde{\boldsymbol{\nu}}_2) = d^{(\mathrm{i})}(\tilde{\boldsymbol{\nu}}_1, \tilde{\boldsymbol{\nu}}_2) + a^{(\mathrm{i})}\overline{d}^{(\mathrm{i})}(\nabla\tilde{\boldsymbol{\nu}}_1, \nabla\tilde{\boldsymbol{\nu}}_2), \tag{12}$$

$$\tilde{d}^{(\mathrm{c})}(\tilde{\boldsymbol{\nu}}_1, \tilde{\boldsymbol{\nu}}_2) = d^{(\mathrm{c})}(\tilde{\boldsymbol{\nu}}_1, \tilde{\boldsymbol{\nu}}_2) + a^{(\mathrm{c})}\overline{d}^{(\mathrm{c})}(\nabla\tilde{\boldsymbol{\nu}}_1, \nabla\tilde{\boldsymbol{\nu}}_2), \tag{13}$$

$$\tilde{d}^{(\mathrm{s})}(\tilde{\boldsymbol{\nu}}_1, \tilde{\boldsymbol{\nu}}_2) = d^{(\mathrm{s})}(\tilde{\boldsymbol{\nu}}_1, \tilde{\boldsymbol{\nu}}_2) + a^{(\mathrm{s})}\overline{d}^{(\mathrm{s})}(\nabla\tilde{\boldsymbol{\nu}}_1, \nabla\tilde{\boldsymbol{\nu}}_2), \tag{14}$$

where $a^{(\mathrm{i})}$, $a^{(\mathrm{c})}$ and $a^{(\mathrm{s})}$ are constants introduced to adjust physical dimensions; they are of the type $(length)^2$.

**Remark 8.** When $\mathcal{M}$ is per se a linear space or for reasons of modeling is embedded isometrically in a linear space, the results in [2] and [18] apply directly to characterize the topological properties of the space $\mathcal{C}$.

### *2.2. Something more about kinematics*

Motions are sufficiently smooth curves over $\mathcal{C}$. For a given interval of time $[0, \overline{t}]$, we then have mappings $[0, \overline{t}] \ni t \longmapsto (\tilde{\mathbf{x}}_t, \tilde{\boldsymbol{\nu}}_t) \in \mathcal{C}$ and indicate with $\mathbf{x} = \tilde{\mathbf{x}}(\mathbf{X}, t)$ and $\boldsymbol{\nu} = \tilde{\boldsymbol{\nu}}(\mathbf{X}, t)$ the current place at $t$ of a material element resting at $\mathbf{X}$ when $t = 0$ and the current value of the order parameter.

With $\dot{\mathbf{x}}$ and $\dot{\boldsymbol{\nu}}$ we denote rates given by $(\mathrm{d}\tilde{\mathbf{x}}/\mathrm{d}t)(\mathbf{X}, t)$ and $(\mathrm{d}\tilde{\boldsymbol{\nu}}/\mathrm{d}t)(\mathbf{X}, t)$ respectively, with $\dot{\boldsymbol{\nu}} \in T_{\tilde{\boldsymbol{\nu}}(\mathbf{X},t)}\mathcal{M}$. They have counterparts $\mathbf{v}$ and $\boldsymbol{\upsilon}$ in the current place $\mathcal{B}$ given by

$$\mathcal{B}\times[0,\overline{t}] \ni (\mathbf{x}, t) \overset{\tilde{\mathbf{v}}}{\longmapsto} \mathbf{v} = \tilde{\mathbf{v}}(\mathbf{x}, t) \in T_{\mathbf{x}}\mathcal{B} \tag{15}$$

and

$$\mathcal{B}\times[0,\overline{t}] \ni (\mathbf{x}, t) \overset{\tilde{\boldsymbol{\upsilon}}}{\longmapsto} \boldsymbol{\upsilon} = \tilde{\boldsymbol{\upsilon}}(\mathbf{x}, t) \in T_{\tilde{\boldsymbol{\nu}}(\mathbf{x}(\mathbf{X},t),t)}\mathcal{M}, \tag{16}$$

obtained through the mapping $\mathbf{X} \longmapsto \mathbf{x} = \tilde{\mathbf{x}}(\mathbf{X})$ at each $t$. We have $\dot{\mathbf{x}} = \mathbf{v}$ but $\boldsymbol{\upsilon} = \dot{\boldsymbol{\nu}} + (\text{grad}\,\boldsymbol{\nu})\mathbf{v}$ and we may write also $\boldsymbol{\upsilon} = \dot{\boldsymbol{\nu}} + (\nabla\boldsymbol{\nu})\mathbf{F}^{-1}\mathbf{v} = \dot{\boldsymbol{\nu}} - (\nabla\boldsymbol{\nu})\dot{\mathbf{X}}$, where $\dot{\mathbf{X}}$ is the material velocity $-\mathbf{F}^{-1}\mathbf{v}$ associated with the inverse mapping $\mathbf{X} = \tilde{\mathbf{x}}^{-1}(\mathbf{x}, t)$. For the acceleration of the order parameter we have

$$\ddot{\nu}^{\alpha} = \partial_t \dot{\nu}^{\alpha} + \check{\Gamma}^{\alpha}_{\beta\gamma}\dot{\nu}^{\beta}\dot{\nu}^{\gamma} \tag{17}$$

with

$$\check{\Gamma}^{\alpha}_{\beta\gamma} = \tfrac{1}{2}\mathbf{g}^{\alpha\delta}_{\mathcal{M}}(\partial_{\nu^{\gamma}}(\mathbf{g}_{\mathcal{M}})_{\beta\delta} + \partial_{\nu^{\beta}}(\mathbf{g}_{\mathcal{M}})_{\gamma\delta} - \partial_{\nu^{\delta}}(\mathbf{g}_{\mathcal{M}})_{\beta\gamma}) \tag{18}$$

Christoffel symbols relevant for $\mathcal{M}$. The expression (17) enters the representation of possible inertial terms pertaining to the substructure.[2]

Let $\mathbf{g}$ be the metric in the ambient space (in general, for curved frames we have $\mathbf{x} \longmapsto \mathbf{g}(\mathbf{x})$, i.e. $\mathbf{g}$ depends on the place), by indicating with $\mathbf{F}^{\mathrm{T}}$ the transpose of $\mathbf{F}$, the mapping

$$\mathcal{B}_0 \ni \mathbf{X} \longmapsto (\mathbf{F}^{\mathrm{T}}\mathbf{F}) \equiv \mathbf{C}(\mathbf{X}) \in \mathrm{Sym}^{+}(T_{\mathbf{X}}\mathcal{B}_0, T^{*}_{\mathbf{X}}\mathcal{B}_0) \tag{19}$$

is the pull-back of $\mathbf{g}$ through the deformation $\tilde{\mathbf{x}}$ and in coordinates we have $C_{AB} = (F^{\mathrm{T}})^{i}_{A}\, g_{ij} F^{j}_{B}$.

If $\boldsymbol{\gamma}$ is the 'material' metric (even flat) in $\mathcal{B}_0$ (i.e. $\mathcal{B}_0 \ni \mathbf{X} \overset{\tilde{\boldsymbol{\gamma}}}{\longmapsto} \boldsymbol{\gamma} = \tilde{\boldsymbol{\gamma}}(\mathbf{X}) \in \mathrm{Sym}^{+}(\mathrm{Vec}, \mathrm{Vec})$), the difference $(\mathbf{C} - \boldsymbol{\gamma})(\mathbf{X})$ is twice the non-linear deformation tensor $\mathbf{E}(\mathbf{X})$ measuring relative changes of lengths by using $\mathcal{B}_0$ as paragon setting. In an alternative point of view, we may consider $\mathcal{B}$ as paragon setting pushing forward $\boldsymbol{\gamma}$ and comparing lengths there as explained in all treatises on non-linear elasticity of simple bodies in chapters dealing with measures of deformation.

In the case of complex bodies the matter may be more intricate and a general (in certain sense abstract) treatment of measures of deformation seems to be absent (the only exception being the basic remarks in [5]). The key point is the specific nature of $\boldsymbol{\nu}$. In fact, when $\boldsymbol{\nu}$ represents, e.g., a microdisplacement, an independent rotation or an independent deformation, its gradient enters the measures of deformation together with $\boldsymbol{\nu}$ itself. On the contrary, when $\boldsymbol{\nu}$ describes a property not related strictly with changes of lengths (say in the case in which $\boldsymbol{\nu}$ represents the volume fraction of a phase in a two-phase material or the spontaneous polarization in ferroelectrics, etc.), standard $\mathbf{C}$ and $\mathbf{E}$, or their spatial counterparts, are sufficient to measure the macroscopic deformation.

In general, we could imagine to have a map of the type

$$(\mathbf{F}, \mathbf{g}, \boldsymbol{\nu}, \nabla\boldsymbol{\nu}) \longmapsto \mathbf{G}(\mathbf{F}, \mathbf{g}, \boldsymbol{\nu}, \nabla\boldsymbol{\nu}) \in \mathrm{Sym}^{+}(T_{\mathbf{X}}\mathcal{B}_0, T^{*}_{\mathbf{X}}\mathcal{B}_0) \tag{20}$$

with $\mathbf{G}$ a metric in $\mathcal{B}_0$ involving the pull-back of $\mathbf{g}$, and to define a deformation tensor $\bar{\mathbf{E}}(\mathbf{X})$ as $(1/2)(\mathbf{G} - \boldsymbol{\gamma})(\mathbf{X})$, erasing $\boldsymbol{\nu}$ and/or $\nabla\boldsymbol{\nu}$ each time in which circumstances suggest such a cancellation.

Of course, the information furnished by (20) is rather volatile unless we put it in a context specifying its nature for some particular substructure.

**Remark 9.** Cosserat [10] and micromorphic [28] materials are well-known classes of complex bodies in which the order parameter (or its gradient) affects the representation of

[2] The influence of $\check{\boldsymbol{\gamma}}$ on inertia is discussed in [7].

the measures of deformation (see also essential remarks in [4]). In the former class each material element is pictured as a 'small' rigid body that may undergo rotations independent of the surrounding material elements. Of course, neighboring relative rotations may alter lengths, in a sense described e.g. in [10] or [4]. In the latter class one imagines that each material element may suffer independent deformations (it is like a ball of rubber). Measures of relative deformations may be then introduced. Less popular is the case of microcracked bodies and we give some details about it in order to construct an explicit example of the possible influence of the order parameter and its gradient on the measures of deformation. In fact, when microcracks are smeared throughout a body, the material element is pictured as a 'patch' of matter endowed with a population of microcracks that can be considered either as planar sharp defects not interpenetrated by interatomic bonds or as elliptic voids with non-null volume and one dimension very small with respect to the others. If we consider *frozen* the microcracks in a given material element placed at $\mathbf{X}$ in $\mathcal{B}_0$, a standard deformation $\tilde{\mathbf{x}}$ puts it (or better its centre of mass) in the place $\mathbf{x} = \tilde{\mathbf{x}}(\mathbf{X})$. Now, if we allow the microcracks to deform (say without growing irreversibly for the sake of simplicity), the centre of mass of each material element undergoes in principle a shift[3] toward a new place $\mathbf{x}'$. If we indicate with $\mathcal{B}'$ the minimal regular region containing the collection of $\mathbf{x}'$, each one corresponding to each $\mathbf{x}$, we may imagine to obtain $\mathcal{B}'$ from $\mathcal{B}$ by means of a sufficiently smooth mapping $\mathfrak{f}$ such that $\mathbf{x}' = (\mathfrak{f} \circ \tilde{\mathbf{x}})(\mathbf{X})$ and $\mathfrak{f}(\mathcal{B}) = \mathcal{B}'$. By denoting with grad the gradient with respect to $\mathbf{x}$, as before, by chain rule we get $\nabla(\mathfrak{f} \circ \tilde{\mathbf{x}})(\mathbf{X}) = ((\operatorname{grad} \mathfrak{f})\nabla\tilde{\mathbf{x}})(\mathbf{X}) = \mathbf{F}^{(\mathrm{m})}\mathbf{F}$, where $\mathbf{F}^{(\mathrm{m})} = (\operatorname{grad} \mathfrak{f})(\mathbf{x})$ is the gradient of deformation from $\mathcal{B}$ to $\mathcal{B}'$, i.e., $\mathbf{F}^{(\mathrm{m})} = \operatorname{Hom}(T_{\mathbf{x}}\mathcal{B}, T_{\mathbf{x}'}\mathcal{B}')$. If we indicate with $\mathsf{d}(\mathbf{X}) = (\mathfrak{f} \circ \tilde{\mathbf{x}})(\mathbf{X}) - \tilde{\mathbf{x}}(\mathbf{X})$ the displacement from $\mathbf{x}$ to $\mathbf{x}'$, defined as a field over $\mathcal{B}_0$, since $\nabla\mathsf{d}(\mathbf{X}) = (\operatorname{grad} \mathsf{d}_a)\mathbf{F}$, with $\mathsf{d}_a = \mathsf{d} \circ \tilde{\mathbf{x}}^{-1}$, we get the additive decomposition $\nabla(\mathfrak{f} \circ \tilde{\mathbf{x}})(\mathbf{X}) = \mathbf{F} + \nabla\mathsf{d}(\mathbf{X})$. If we write $\mathbf{F}_{\mathrm{tot}} = \mathbf{F} + \nabla\mathsf{d}(\mathbf{X})$, $\mathbf{F}_{\mathrm{tot}}$ is the gradient of deformation from $\mathcal{B}_0$ to $\mathcal{B}'$. So that the right Cauchy-Green tensor $\mathbf{C}_{\mathrm{tot}}(\mathbf{X}) = \mathbf{F}_{\mathrm{tot}}^{\mathrm{T}}\mathbf{F}_{\mathrm{tot}}$ involves $\nabla\mathsf{d}$. Moreover, by comparing the additive decomposition of $\mathbf{F}_{\mathrm{tot}}$ with the multiplicative one, namely $\mathbf{F}^{(\mathrm{m})}\mathbf{F}$, we realize that $\mathbf{F}^{(\mathrm{m})} = \mathbf{I} + \nabla\mathsf{d}(\mathbf{X})\mathbf{F}^{-1} = \mathbf{I} + \operatorname{grad} \mathsf{d}_a$. The direct (perhaps naive) description of the kinematics of microcracked bodies just sketched here follows [21,26]; however, it can be obtained by using the procedure involving the limit of bodies described in [12,13].

### 2.3. Observers

Three geometrical environments are necessary to describe the motion of a complex body: the time interval $[0, \bar{t}]$, the ambient space $\mathcal{E}^3$ (of course $\mathcal{E}^2$ or $\mathcal{E}^1$ if we deal with two- or one-dimensional bodies) and $\mathcal{M}$.

In classical mechanics an observer is a *representation* of the measure of time and the ambient space so that changes of observers are changes of representation.

Here, our point of view is that the notion of observer should involve all the descriptors of the morphology of the body and its motion. Then, for us, *an observer $\mathcal{O}$ is a representation* of (i) the interval of time, (ii) the ambient space $\mathcal{E}^3$ and (iii) the manifold $\mathcal{M}$ of substructural shapes [25].

[3] Here the circumstance that there is a population of microcracks in the material element and that each microcrack is not a spheric void is crucial.

Such a definition is a natural extension of the standard point of view including just (i) and (ii): it takes into account the presence of the manifold $\mathcal{M}$ of substructural shapes as natural geometric ingredient in the representation of the body.

For the sake of simplicity we shall consider observers coinciding about the measure of time.

General changes of observers, as used below, will then involve automorphism of $\mathcal{E}^3$ and the action of arbitrary Lie groups over $\mathcal{M}$. In this sense the notion of change of observers involves the pair of one-parameter families of transformations indicated below.

A1. $\mathbb{R}^+ \ni s_2 \longmapsto \mathbf{f}^2_{s_2} \in \mathrm{Aut}(\mathcal{E}^3)$, with $\mathbf{f}^2_0$ the identity. [4] We assume that each $\mathbf{f}^2$ is sufficiently smooth and put $\mathbf{f}^{2\prime}_0(\mathbf{x}) = \mathbf{v}$, where the prime denotes differentiation with respect to $s_2$.

A2. A Lie group $G$, with Lie algebra $\mathfrak{g}$, acts over $\mathcal{M}$. If $\xi \in \mathfrak{g}$, its action over $\boldsymbol{\nu} \in \mathcal{M}$ is indicated with $\xi_{\mathcal{M}}(\boldsymbol{\nu})$. By indicating with $\boldsymbol{\nu}_g$ the value of $\boldsymbol{\nu}$ after the action of $g \in G$, if we consider a one-parameter smooth curve $\mathbb{R}^+ \ni s_3 \longmapsto g_{s_3} \in G$ over $G$, such that $\xi = (\mathrm{d}g_{s_3}/\mathrm{d}s_3)|_{s_3=0}$, and its corresponding curve $s_3 \longmapsto \boldsymbol{\nu}_{g_{s_3}}$ over $\mathcal{M}$, starting from a given $\boldsymbol{\nu}$, we have $\xi_{\mathcal{M}}(\boldsymbol{\nu}) = (\mathrm{d}/\mathrm{d}s_3)\boldsymbol{\nu}_{g_{s_3}}|_{s_3=0}$.

We specify for future use in special cases the relations characterizing 'rigid' changes of observers, i.e. the ones characterized by the contemporary action of SO(3) on $\mathcal{M}$ and $\mathcal{E}^3$. The latter action is induced by standard isometric transformations of point space.

Let $[0, \overline{t}] \ni t \longmapsto \mathbf{Q}(t) \in \mathrm{SO}(3)$, with $\mathbf{Q}(0) = \mathrm{Id}_{\mathrm{SO}(3)}$, Id the identity, be a smooth curve on SO(3). Let also $\mathcal{O}$ be an assigned representation of $\mathcal{E}^3$ at $t = 0$. At any $t \neq 0$ we may associate an observer $\mathcal{O}'_t$ obtained from $\mathcal{O}$ by means of $\mathbf{Q}(t)$. The transformation from $\mathcal{O}$ to $\mathcal{O}'_t$ is isometric, then a point $\mathbf{x}$ seen by $\mathcal{O}$ is mapped in a point $\mathbf{x}' = \mathbf{w}(t) + \mathbf{Q}(t)(\mathbf{x} - \mathbf{x}_0)$, where $t \longmapsto \mathbf{w}(t)$ is an arbitrary point valued function smooth in time and $\mathbf{x}_0$ a fixed point chosen at will in space. If we calculate the rate of $\mathbf{x}'$ and pull-back it in $\mathcal{O}$ writing $\dot{\mathbf{x}}^*$ for $\mathbf{Q}^{\mathrm{T}}\dot{\mathbf{x}}'$, we get the standard relation

$$\dot{\mathbf{x}}^* = \dot{\mathbf{x}} + c(t) + \dot{\mathbf{q}} \wedge (\mathbf{x} - \mathbf{x}_0), \tag{21}$$

characterizing a classical change of observer, where $\mathbf{c} = \mathbf{Q}^{\mathrm{T}}\mathbf{w}$ is the translational velocity and $\dot{\mathbf{q}}$ the rotational one (it can be considered as the action of SE(3) over Vec, the affine translation space over $\mathcal{E}^3$). If the element $\dot{\mathbf{q}}\wedge$ of the Lie algebra $\mathfrak{so}(3)$, selected arbitrarily in (21), acts also on $\mathcal{M}$, at each $\boldsymbol{\nu}$ it induces a rate (that we call 'rigid', with the subscript $R$, just for reminding the circumstance that it is related to a spatial rigid body motion) given by $\mathcal{A}\dot{\mathbf{q}} \in T_{\boldsymbol{\nu}}\mathcal{M}$, where $\mathcal{A}(\boldsymbol{\nu}) \in \mathrm{Hom}(\mathrm{Vec}, T_{\boldsymbol{\nu}}\mathcal{M})$. If $\boldsymbol{\nu}_q$ denotes the value of $\boldsymbol{\nu}$ after the action of SO(3) over $\mathcal{M}$, we have $\mathcal{A} = (\mathrm{d}\boldsymbol{\nu}_q/\mathrm{d}\mathbf{q})|_{\mathbf{q}=0}$, where $\mathbf{q}$ is a vector connected with $\mathbf{Q} \in \mathrm{SO}(3)$ by the formula $\mathbf{Q} = \exp(-\mathsf{e}\mathbf{q})$, with $\mathsf{e}$ Ricci's permutation index. For example, let $\mathcal{M}$ be coincident with $S^2$ as in the case of magnetostrictive materials). For any $\boldsymbol{\tau} \in S^2$, we get $\boldsymbol{\tau}_{\mathbf{q}} = \mathbf{Q}\tau$, then $\mathcal{A} = -\boldsymbol{\tau}\wedge$. Then the corresponding change in observer corresponding to (21) is given by

$$\dot{\boldsymbol{\nu}}^* = \dot{\boldsymbol{\nu}} + \mathcal{A}\dot{\mathbf{q}}. \tag{22}$$

[4] $\mathrm{Aut}(\mathcal{E}^3)$ is the group of automorphisms of $\mathcal{E}^3$.

In a multifield setting, *classical* changes in observers coinciding about the measure of time are then given by (21) and (22).

### 2.4. Relabeling

Another transformation playing a role below is the *relabeling* of material elements in the reference place $\mathcal{B}_0$. With the term 'relabeling' we indicate a generic transformation of $\mathcal{B}_0$ induced by a $C^1(\mathcal{B}_0)$ point valued mapping $\mathbf{f}^1$ such that (i) it is *isocoric* (i.e. the volume is preserved) and (ii) establishes also a diffeomorphism between $\mathcal{B}_0$ and $\mathbf{f}^1(\mathcal{B}_0)$. From a physical point of view, the important part of the action of $\mathbf{f}^1$ relies in a sort of 'permutation' of possible defects or, better, inhomogeneities in $\mathcal{B}_0$. In particular, we will consider a one-parameter family of such isocoric diffeomorphisms defined below:

A3. $\mathbb{R}^+ \ni s_1 \longmapsto \mathbf{f}^1_{s_1} \in S\,\mathrm{Diff}(\mathcal{B}_0)$ is a smooth curve with $\mathbf{f}^1_0$ the identity; at each $s_1$ we get $\mathbf{X} \longmapsto \mathbf{f}^1_{s_1}(\mathbf{X})$, with $\mathrm{Div}\,\mathbf{f}^{1\prime}_{s_1}(\mathbf{X}) = 0$, where the prime denotes differentiation with respect to the parameter $s_1$. We put $\mathbf{f}^{1\prime}_0(\mathbf{X}) = \mathbf{w}$.

## 3. Lagrangian 3+1 forms and balance equations

The multifield theoretical analogue of the theory of elasticity for complex bodies (i.e., bodies with material substructure) relies on rather articulated fiber bundles.

We start by considering a fiber bundle $\pi : \mathcal{Y} \to \mathcal{B}_0 \times [0, \bar{t}]$ such that $\pi^{-1}(\mathbf{X}, t) = \mathcal{E}^3 \times \mathcal{M}$. A generic section $\eta \in \Gamma(\mathcal{Y})$ of $\mathcal{Y}$ is then $\eta(\mathbf{X}, t) = (\mathbf{X}, t, \mathbf{x}, \boldsymbol{\nu})$. For sufficiently smooth sections, the first jet bundle $J^1\mathcal{Y}$ over $\mathcal{Y}$ is such that $J^1\mathcal{Y} \ni j^1(\eta)(\mathbf{X}, t) = (\mathbf{X}, t, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{F}, \boldsymbol{\nu}, \dot{\boldsymbol{\nu}}, \nabla\boldsymbol{\nu})$.

Up to this point the discussion has been purely geometric. No issues related with the constitutive nature of the body and with the interactions arising inside it have been discussed. They enter into play here: we assume that the general body under examination is made of (non-linear) elastic material. In this case we may associate with it the canonical Lagrangian $3 + 1$ form

$$L : J^1\mathcal{Y} \to \wedge^{3+1}(\mathcal{B}_0 \times [0, \bar{t}]). \tag{23}$$

The definition of the elements of the space of $3 + 1$ forms $\wedge^{3+1}(\mathcal{B}_0 \times [0, \bar{t}])$ would require some care. In fact, $\mathcal{B}_0 \times [0, \bar{t}]$ is a manifold with boundary coinciding only with $\{0\} \times \mathcal{B}_0 \cup \{\bar{t}\} \times \mathcal{B}_0$ (because $\mathcal{B}_0$ is open and coincides with the interior of its closure), while the definition of odd forms is immediate on manifolds without boundary. However, one is commonly interested in evaluating the variation of the total Lagrangian

$$\bar{L}(\mathcal{B}_0) = \int_{\mathcal{B}_0 \times [0,\bar{t}]} L(j^1(\eta)(\mathbf{X}, t)); \tag{24}$$

so that, in defining $L(j^1(\eta))$, possible problems related with boundary points do not play any role. $L$ admits the structure

$$L(j^1(\eta)(\mathbf{X}, t)) = \mathcal{L}(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{F}, \boldsymbol{\nu}, \dot{\boldsymbol{\nu}}, \nabla\boldsymbol{\nu})\,\mathrm{d}^3\mathbf{X} \wedge \mathrm{d}\,t \tag{25}$$

with

$$\mathcal{L}(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{F}, \boldsymbol{\nu}, \dot{\boldsymbol{\nu}}, \nabla\boldsymbol{\nu}) = \tfrac{1}{2}\rho_0 |\dot{\mathbf{x}}|^2 + \rho_0\chi(\boldsymbol{\nu}, \dot{\boldsymbol{\nu}}) - \rho_0 e(\mathbf{X}, \mathbf{F}, \boldsymbol{\nu}, \nabla\boldsymbol{\nu}) - \rho_0 w(\mathbf{x}, \boldsymbol{\nu}), \tag{26}$$

where $\rho_0$ is the referential mass density (conserved during the motion), $\chi$ the substructural *kinetic co-energy* (needed when physical circumstances prescribe substructural inertia), $e$ the elastic energy density and $w$ the density of the potential of external actions, all per unit mass. The presence of $\boldsymbol{\nu}$ in the list of entries of $\chi$ and $e$ is due to the non-trivial structure of $\mathcal{M}$. In general the elements of $T\mathcal{M}$ cannot be separated invariantly unless a parallelism can be found over $\mathcal{M}$. From classical non-linear field theories we know that $\mathbf{x}$ disappears from the list of entries of $e$ due to reasons of invariance, but $\boldsymbol{\nu}$ does not (see [4]).

We assume that $\mathcal{L}(\cdot)$ be sufficiently smooth so that we may find at least one section $\eta$ satisfying Euler–Lagrange equations

$$\overset{\cdot}{\partial_{\dot{\mathbf{x}}}\mathcal{L}} = \partial_{\mathbf{x}}\mathcal{L} - \operatorname{Div}\partial_{\mathbf{F}}\mathcal{L}, \tag{27}$$

$$\overset{\cdot}{\partial_{\dot{\boldsymbol{\nu}}}\mathcal{L}} = \partial_{\boldsymbol{\nu}}\mathcal{L} - \operatorname{Div}\partial_{\nabla\boldsymbol{\nu}}\mathcal{L}. \tag{28}$$

With respect to the families of transformations characterizing relabeling and changes of observers, for the sake of brevity, we indicate with $\mathbf{f}^1, \mathbf{f}^2$ and $\boldsymbol{\nu}_g$ the values $\mathbf{f}^1_{s_1}(\mathbf{X}), \mathbf{f}^2_{s_2}(\mathbf{x}), \boldsymbol{\nu}_{g_{s_3}}(\mathbf{X})$.

**Definition 1.** $\mathcal{L}$ is invariant with respect to the action of $\mathbf{f}^1_{s_1}$, $\mathbf{f}^2_{s_2}$ and $G$ if

$$\begin{aligned}&\mathcal{L}(\mathbf{X}, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{F}, \boldsymbol{\nu}, \dot{\boldsymbol{\nu}}, \nabla\boldsymbol{\nu})\\ &\quad = \mathcal{L}(\mathbf{f}^1, \mathbf{f}^2, (\operatorname{grad}\mathbf{f}^2)\dot{\mathbf{x}}, (\operatorname{grad}\mathbf{f}^2)\mathbf{F}(\nabla\mathbf{f}^1)^{-1}, \boldsymbol{\nu}_g, \dot{\boldsymbol{\nu}}_g, (\nabla\boldsymbol{\nu}_g)(\nabla\mathbf{f}^1)^{-1}),\end{aligned} \tag{29}$$

for any $g \in G$ and $s_1, s_2 \in \mathbb{R}^+$.

Let $\mathcal{Q}$ and $\mathcal{F}$ be *scalar* and *vector* densities given respectively by

$$\mathcal{Q} = \partial_{\dot{\mathbf{x}}}\mathcal{L} \cdot (\mathbf{v} - \mathbf{F}\mathbf{w}) + \partial_{\dot{\boldsymbol{\nu}}}\mathcal{L} \cdot (\xi_{\mathcal{M}}(\boldsymbol{\nu}) - (\nabla\boldsymbol{\nu})\mathbf{w}), \tag{30}$$

$$\mathcal{F} = \mathcal{L}\mathbf{w} + (\partial_{\mathbf{F}}\mathcal{L})^{\mathrm{T}}(\mathbf{v} - \mathbf{F}\mathbf{w}) + (\partial_{\nabla\boldsymbol{\nu}}\mathcal{L})^*(\xi_{\mathcal{M}}(\boldsymbol{\nu}) - (\nabla\boldsymbol{\nu})\mathbf{w}), \tag{31}$$

where $(\partial_{\nabla\boldsymbol{\nu}}\mathcal{L})^* \in \operatorname{Hom}(T_{\boldsymbol{\nu}}\mathcal{M}, T_{\mathbf{X}}\mathcal{B}_0)$ and $(\partial_{\mathbf{F}}\mathcal{L})^{\mathrm{T}} \in \operatorname{Hom}(T_{\mathbf{x}}\mathcal{B}, T_{\mathbf{X}}\mathcal{B}_0)$.

**Theorem 1** (Capriz and Mariano [9]). *If $\mathcal{L}$ is invariant under $\mathbf{f}^1_{s_1}$, $\mathbf{f}^2_{s_2}$ and $G$, for any $g \in G$ and $s_1, s_2 \in \mathbb{R}^+$, then*

$$\dot{\mathcal{Q}} + \operatorname{Div}\mathcal{F} = 0. \tag{32}$$

The detailed proof is contained in [9]. Here we remind only that it is based on a direct explicit calculation of the terms in (32) on the basis of (30) and (31), and on the exploitation of the relations

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s_1}\mathcal{L}|_{s_1=0,s_2=0,s_3=0} = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}s_2}\mathcal{L}|_{s_1=0,s_2=0,s_3=0} = 0,\\ &\frac{\mathrm{d}}{\mathrm{d}s_3}\mathcal{L}|_{s_1=0,s_2=0,s_3=0} = 0,\end{aligned} \tag{33}$$

which are consequences of the request of invariance for $\mathcal{L}$.

The following corollaries hold true for any Lagrangian density of the type (26) which is invariant in the sense of Definition 1 [9]:

**Corollary 1.** *If* $\mathbf{f}^2_{s_2}$ *alone acts on* $\mathcal{L}$ *leaving* $\mathbf{v}$ *arbitrary, from* (32) *we get Cauchy balance of momentum*

$$\rho_0\ddot{\mathbf{x}} = \rho_0\mathbf{b} + \operatorname{Div}\mathbf{P}, \tag{34}$$

*where* $\mathbf{P} = -\partial_{\mathbf{F}}\mathcal{L} \in \operatorname{Hom}(T^*_{\mathbf{X}}\mathcal{B}_0, T^*_{\mathbf{x}}\mathcal{B})$ *is the first Piola–Kirchhoff stress and* $\rho_0\mathbf{b} = \partial_{\mathbf{x}}\mathcal{L} \in T^*_{\mathbf{x}}\mathcal{B}$ *the vector of body forces*.

**Corollary 2.** *If* $G$ *arbitrary acts alone on* $\mathcal{L}$, *from* (32) *and the arbitrariness of the element* $\xi$ *of the Lie algebra of* $G$ (*chosen to define* $\xi_{\mathcal{M}}(\boldsymbol{v})$) *we get Capriz balance of substructural interactions*

$$\rho_0(\dot{\overline{\partial_{\dot{\boldsymbol{v}}}\chi}} - \partial_{\boldsymbol{v}}\chi) = -\mathbf{z} + \rho_0\boldsymbol{\beta}^{ni} + \operatorname{Div}\mathcal{S}, \tag{35}$$

*in covariant way, where* $\rho_0\boldsymbol{\beta}^{ni} = -\rho_0\partial_{\boldsymbol{v}}w \in T^*_{\boldsymbol{v}}\mathcal{M}$ *represents bulk non-inertial external interactions acting on the substructure,* $\mathcal{S} = -\partial_{\nabla\boldsymbol{v}}\mathcal{L} \in \operatorname{Hom}(T^*_{\mathbf{X}}\mathcal{B}_0, T^*_{\boldsymbol{v}}\mathcal{M})$ *takes into account contact substructural interactions between neighboring material elements* (*and is called microstress*) *and* $\mathbf{z} = -\rho_0\partial_{\boldsymbol{v}}e \in T^*_{\boldsymbol{v}}\mathcal{M}$ *indicates self-interactions of the substructure in each material element* (*and is called self-force*).

**Corollary 3.** *Let* $G = \mathrm{SO}(3)$ *and, for any element* $\dot{\mathbf{q}}\wedge$ *of its Lie algebra,* $\mathbf{f}^2_{s_2}$ *be such that* $\mathbf{v} = \dot{\mathbf{q}} \wedge (\mathbf{x} - \mathbf{x}_0)$ *with* $\mathbf{x}_0$ *a fixed point in space. In other words, we require that the same copy of* SO(3) *acts both on the ambient space and on* $\mathcal{M}$. *If we require the invariance of* $e$ *with respect to the action of the special choices of* $G$ *and* $\mathbf{f}^2$ *just mentioned, we obtain*

$$\operatorname{skw}(\partial_{\mathbf{F}}e\mathbf{F}^{\mathrm{T}}) = \mathsf{e}(\mathcal{A}^{\mathrm{T}}\partial_{\boldsymbol{v}}e + (\nabla\mathcal{A}^{\mathrm{T}})^t\partial_{\nabla\boldsymbol{v}}e), \tag{36}$$

*where* $\mathsf{e}$ *is Ricci's alternator and* $\operatorname{skw}(\cdot)$ *extracts the skew-symmetric part of its argument. The previous statement render more perspicuous* Remark 3 *of* [9].

**Corollary 4.** *If* $\mathbf{f}^1_{s_1}$ *alone acts on* $\mathcal{L}$, *with* $\mathbf{w}$ *arbitrary, from* (32) *we get*

$$\dot{\overline{(\mathbf{F}^{\mathrm{T}}\partial_{\dot{\mathbf{x}}}\mathcal{L} + (\nabla\boldsymbol{v})^*\partial_{\dot{\boldsymbol{v}}}\mathcal{L})}} - \operatorname{Div}(\mathbb{P} - (\tfrac{1}{2}\rho_0|\dot{\mathbf{x}}|^2 + \rho_0\chi(\boldsymbol{v}, \dot{\boldsymbol{v}}))\mathbf{I}) - \partial_{\mathbf{X}}\mathcal{L} = \mathbf{0}, \tag{37}$$

*where* $\mathbb{P} = \rho_0 e\mathbf{I} - \mathbf{F}^{\mathrm{T}}\mathbf{P} - (\nabla\boldsymbol{v})^*\mathcal{S} \in \operatorname{Aut}(T^*_{\mathbf{X}}\mathcal{B}_0)$, *with* $\mathbf{I}$ *the second order unit tensor, is the modified Eshelby tensor for continua with substructure derived in the general setting in* [22,23]. *In particular,* (37) *coincides with the balance of configurational forces for a continuum with substructure in absence of dissipative internal forces driving defects. The balance* (37), *in fact, is only a consequence of the invariance with respect to relabeling.*

**Corollary 5.** *Let* $G = \mathrm{SO}(3)$ *and, for any element* $\dot{\mathrm{q}}\wedge$ *of its Lie algebra,* $\mathbf{f}^1_{s_1}$ *be also such that* $\mathbf{w} = \dot{\mathrm{q}} \wedge (\mathbf{X} - \mathbf{X}_0)$ *with* $\mathbf{X}_0$ *a fixed point in* $\mathcal{B}_0$. *If the material is homogeneous, and only such special choices of* $\mathbf{f}^1_{s_1}$ *and* $G$ act on $\mathcal{L}$, $\mathbb{P}$ *is symmetric*.

**Remark 10.** It may be asked whether or not an integral version of (35) can be postulated as integral balance principle of substructural interactions and then used as a first principle. In general the answer is negative unless $\mathcal{M}$ is a linear space or is embedded in a linear space.

The reason relies upon the circumstance that the eventual integrands $\boldsymbol{\beta}$, $\mathbf{z}$ and $\mathcal{S}\mathbf{n}$ would take values in $T^*\mathcal{M}$ which is a non-linear space unless the above mentioned situations of linearity for $\mathcal{M}$ occur, so the integrals are not defined. In other words, each time we use an integral version of (35) we presume implicitly an embedding in a linear space; on the contrary it does not make sense.

## 4. Discontinuity surfaces

In common special cases, solutions of (27) and (28) or, with other notations, of (34) and (35) are not smooth and may display discontinuities concentrated on submanifolds of codimension 1. Moreover, experiments display domain formation and branching of microstructures of various nature (see e.g. cases of nematic order in liquid crystals, polarization in ferroelectrics, magnetization in micromagnetics, superconducting domains, etc.).

The presence of the gradient of the morphological descriptor in the list of entries of the energy accounts for the presence of interfaces of domain walls in a smeared sense. However, one may ask what happens when additional discontinuity surfaces occur and there is *interaction between smeared and sharp interfaces*. As an example, the us consider the solidification of a two-phase flow [24]. Two types of interfaces occur in this case: interfaces between the two phases of the fluid that interact with the interface between solid and fluid parts (when it is considered as a sharp interface) and influence its evolution.

Really, even in single phase complex materials the energy may depend on the gradient of the order parameter to account for the inhomogeneous behavior of substructures. For example, in the case of micromorphic materials, each material element is a cell that may undergo micro-deformation independently of the neighboring cells, in addition to the participation to the overall macroscopic deformation [28]. The morphological descriptor is then a second-order symmetric tensor representing such an additional independent deformation. Such a situation may be representative of the behavior of elastomers, for example. Adjacent material elements may then undergo in principle different micro-deformations so that, in going from one element to the other, the energetic landscape changes in accord to the gradient of the micro-deformation and weakly non-local interaction effects of gradient type between neighboring material elements may be accounted for. Even in this case, additional discontinuity surfaces may occur as defects or shock and acceleration waves. Energetic effects associated with the gradient of the micro-deformation may influence them.

Analogous situation occurs also in quasicrystals where collective atomic modes (phason degrees of freedom) occur *within* each crystalline cell (the material element) and are represented by a stretchable vector. The energy depends both on the standard strain and the gradient of such a vector [31]. Even in this case, discontinuity surfaces such as (e.g.) cracks, dislocations or shock waves may intervene. Their behavior is influenced by the energetic effects associated with the gradient of the order parameter (here the vector mentioned above) but they are not strictly modeled by means of it.

The physical examples above allow us to clarify the point of view followed below. *We treat, in fact, situations in which smeared interfaces or gradient effects due to inhomogeneous behavior of material substructures interact with sharp interfaces or, more generally, discontinuity surfaces.*

In particular, we focus our attention on a single discontinuity surface $\Sigma$ defined by

$$\Sigma \equiv \{\mathbf{X} \in cl\mathcal{B}_0,\ f(\mathbf{X}) = 0\}, \tag{38}$$

with $f$ a smooth function with non-singular gradient. It is oriented by the normal vector field $\Sigma \ni \mathbf{X} \overset{\tilde{\mathbf{m}}}{\longmapsto} \mathbf{m} = \tilde{\mathbf{m}}(\mathbf{X}) = \nabla f(\mathbf{X})/|\nabla f(\mathbf{X})|$ and we use the notation $\Pi$ for the second order tensor $\mathbf{I} - \mathbf{m} \otimes \mathbf{m}$.

For any field $\tilde{e}(\cdot)$ continuous and piecewise continuously differentiable on $\Sigma$, we indicate with $\nabla_\Sigma e$ its *surface gradient* at $\mathbf{X}$, with $e = \tilde{e}(\mathbf{X})$, given by $\nabla_\Sigma e = \nabla e \Pi$. The opposite of the surface gradient of $\mathbf{m}$, namely $-\nabla_\Sigma \mathbf{m}$ is indicated with $\mathsf{L}$ and is the curvature tensor.

Let $\mathbf{X} \longmapsto a = \tilde{a}(\mathbf{X})$ be a generic field taking values in a linear space and suffering bounded discontinuities across $\Sigma$. For $\varepsilon > 0$ we indicate with $a^{\pm}$ the limits $\lim_{\varepsilon \to 0} a(\mathbf{X} \pm \varepsilon \mathbf{m})$ which are the outer ($a^+$) and inner ($a^-$) traces of $a$ at $\Sigma$. Then we denote with $[a] = a^+ - a^-$ the jump of $a$ across $\Sigma$ and with $2\langle a \rangle = a^+ + a^-$ its average there, so that for any pair of fields $a_1$ and $a_2$ with the same properties of $a$ we have $[a_1 a_2] = [a_1]\langle a_2 \rangle + \langle a_1 \rangle [a_2]$ if the product $a_1 a_2$ is defined in some way and is distributive.

$\Sigma$ is *coherent* when the jump of the gradient of deformation $\mathbf{F}$ (see Section 2) satisfies the relation $[\mathbf{F}]\Pi = 0$, otherwise it is called *incoherent*. In other words the two pieces of the body separated by the surface do not suffer relative shear.

Moreover, if we attribute any 'virtual' motion to $\Sigma$ by means of a vector field $\Sigma \ni \mathbf{X} \overset{\tilde{\mathbf{u}}}{\longmapsto} \mathbf{u} = \tilde{\mathbf{u}}(\mathbf{X}) \in \mathbb{R}^3$ with normal component $U = \mathbf{u} \cdot \mathbf{m}$ and assume that the velocity $\dot{\mathbf{x}}$ may suffer bounded jumps across $\Sigma$, we get the condition $[\dot{\mathbf{x}}] = -U[\mathbf{F}]\mathbf{m}$, whose proof is textbook affairs.

As essential point we assume also that the morphological descriptor map $\tilde{\boldsymbol{\nu}}$ is *continuous* across $\Sigma$. A special case in which such an assumption plays a prominent role is Landau's theory of phase transitions.

### 4.1. The unstructured case

We treat first the case in which $\Sigma$ is unstructured, i.e. it is free of own surface energy. We derive the balance equations across $\Sigma$ in a way that prove their *covariance* which has been not discussed so far. The procedure we adopt relies upon the exploitation of an integral version of the pointwise relation (32).

We call *part* any regular subset $\mathfrak{b}$ of $\mathcal{B}_0$ with non-vanishing volume measure. We consider an arbitrary part $\mathfrak{b}_\Sigma$ crossing $\Sigma$ in a way in which the intersection of its boundary $\partial\mathfrak{b}_\Sigma$ with $\Sigma$ be a piecewise smooth curve and write for it the integral counterpart of (32), namely

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathfrak{b}_\Sigma} \mathcal{Q}\, \mathrm{d}^3\mathbf{X} + \int_{\partial\mathfrak{b}_\Sigma} \mathfrak{F} \cdot \mathbf{n}\, \mathrm{d}\mathcal{H}^2 = 0, \tag{39}$$

with $\mathrm{d}\mathcal{H}^2$ the two-dimensional Hausdorff measure over $\partial\mathfrak{b}_\Sigma$. If we take fixed the part $\mathfrak{b}_\Sigma$ with respect to the virtual motion of $\Sigma$, by the use of the transport and Gauss theorems (see, e.g. [27,35]), as $\mathfrak{b}_\Sigma \to \mathfrak{b}_\Sigma \cap \Sigma$ we find

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathfrak{b}_\Sigma} \mathcal{Q}\, \mathrm{d}^3\mathbf{X} \to -\int_{\mathfrak{b}_\Sigma \cap \Sigma} [\mathcal{Q}]U\, \mathrm{d}\mathcal{H}^2, \tag{40}$$

$$\int_{\partial \mathfrak{b}_\Sigma} \mathfrak{F} \cdot \mathbf{n} \, \mathrm{d}\mathcal{H}^2 \to \int_{\mathfrak{b}_\Sigma \cap \Sigma} [\mathfrak{F}] \cdot \mathbf{m} \, \mathrm{d}\mathcal{H}^2, \tag{41}$$

so that the arbitrariness of $\mathfrak{b}_\Sigma$ implies the pointwise balance

$$-[\mathcal{Q}]U + [\mathfrak{F}] \cdot \mathbf{m} = 0. \tag{42}$$

**Proposition 1.** *If the transformations A1, A2, A3* (*see* Sections 2.3 *and* 2.4) *are smooth throughout the body, the validity of* (42) *and the invariance of* $\mathcal{L}$ *imply covariant pointwise interfacial balances across* $\Sigma$ *as in the list below*:

1. *The action of* $\mathbf{f}^2_{s_2}$ *alone implies the interfacial balance of standard interactions*

$$[\mathbf{P}]\mathbf{m} = -\rho_0[\dot{\mathbf{x}}]U. \tag{43}$$

2. *The action of G alone implies the interfacial balance of substructural interactions*

$$[\mathcal{S}]\mathbf{m} = -\rho_0[\partial_{\dot{\boldsymbol{\nu}}}\chi]U. \tag{44}$$

3. *The action of* $\mathbf{f}^1_{s_1}$ *alone implies the interfacial configurational balance along the normal* $\mathbf{m}$ *in absence of dissipative forces driving* $\Sigma$, *namely*

$$\mathbf{m} \cdot [\mathbb{P}]\mathbf{m} = \rho_0 U[(\nabla \boldsymbol{\nu})^* \partial_{\boldsymbol{\nu}}\chi] \cdot \mathbf{m} + \tfrac{1}{2}\rho_0[\chi(\boldsymbol{\nu}, \dot{\boldsymbol{\nu}})] - \tfrac{1}{2}\rho_0 U^2[|\mathbf{F}\mathbf{m}|^2]. \tag{45}$$

The proof is obtained by exploiting (42) and is contained in the one of Theorem 2.

Balance equations at unstructured discontinuity surfaces in standard elasticity of simple bodies are derived by means of a direct evaluation of the variation of the total Lagrangian in [15].

### 4.2. The structured case

We consider here $\Sigma$ endowed with a *surface energy density* $\phi$ associated with surface tensions of standard and substructural nature. The presence of $\phi$ models the circumstance that often we have discontinuity thin layers that sustain surface tensions instead of pure surfaces.

Let $\mathbb{F}$ and $\mathbb{N}$ be defined by

$$\mathbb{F} = \langle \mathbf{F} \rangle \Pi, \qquad \mathbb{N} = \langle \nabla \boldsymbol{\nu} \rangle \Pi. \tag{46}$$

They are *surface deformation gradient* and *surface gradient of the morphological descriptor* at $\mathbf{X}$. There exist then two mappings $\tilde{\mathbb{F}}$ and $\tilde{\mathbb{N}}$ such that

$$\Sigma \ni \mathbf{X} \overset{\tilde{\mathbb{F}}}{\longmapsto} \mathbb{F} = \tilde{\mathbb{F}}(\mathbf{X}) \in \mathrm{Hom}(T_{\mathbf{X}}\Sigma, T_{\mathbf{x}}\mathcal{B}), \tag{47}$$

$$\Sigma \ni \mathbf{X} \overset{\tilde{\mathbb{N}}}{\longmapsto} \mathbb{N} = \tilde{\mathbb{N}}(\mathbf{X}) \in \mathrm{Hom}(T_{\mathbf{X}}\Sigma, T_{\boldsymbol{\nu}}\mathcal{M}). \tag{48}$$

Elementary algebra provides us

$$\langle \mathbf{F} \rangle = \mathbb{F} + (\langle \mathbf{F} \rangle \mathbf{m}) \otimes \mathbf{m}, \tag{49}$$

$$\langle \nabla \boldsymbol{\nu} \rangle = \mathbb{N} + (\langle \nabla \boldsymbol{\nu} \rangle \, \mathbf{m}) \otimes \mathbf{m}. \tag{50}$$

The surface energy density is then defined by

$$(\mathbf{m}, \mathbb{F}, \nu, \mathbb{N}) \overset{\tilde{\phi}}{\longmapsto} \phi = \tilde{\phi}(\mathbf{m}, \mathbb{F}, \boldsymbol{\nu}, \mathbb{N}) \tag{51}$$

and we assume that $\tilde{\phi}$ be sufficiently smooth. It is worth noting that the presence of the normal $\mathbf{m}$ in the list of entries of $\tilde{\phi}$ points out that we are considering *anisotropic surfaces*.

We require the *invariance* of $\phi$ with respect to (i) general changes of observers and (ii) relabeling of $\Sigma$.

As discussed above, changes of observers are characterized by the action of the group of automorphisms of $\mathcal{E}^3$ and of a generic Lie group over $\mathcal{M}$. However, definition A3 of $\mathbf{f}^1_{s_1}$ (in Section 2.4) needs to be modified in order to describe the relabeling of $\Sigma$ in addition to the overall relabeling of $\mathcal{B}_0$. Instead of $s_1 \longmapsto \mathbf{f}^1_{s_1}$ we should consider a smooth curve $s_1 \longmapsto \hat{\mathbf{f}}^1_{s_1}$ in $S\,\mathrm{Diff}(\mathcal{B}_0)$ characterized by the properties listed below.

*Relabeling of $\mathcal{B}_0$ including $\Sigma$.*

1. The map $s_1 \longmapsto \hat{\mathbf{f}}^1_{s_1}$ satisfies A3 (in Section 2.4). Moreover, we require that the field $\mathcal{B}_0 \ni \mathbf{X} \longmapsto \mathbf{w} = \tilde{\mathbf{w}}(\mathbf{X}) = \hat{\mathbf{f}}^{1\prime}_0(\mathbf{X})$ is at least of class $C^1(\mathcal{B}_0)$, then it is continuous across $\Sigma$.
2. Each $\hat{\mathbf{f}}^1_{s_1}$ preserves the elements of area of $\Sigma$. Namely, if $\mathrm{d}A$ is the element of area of $\Sigma$ in $\mathcal{B}_0$, $\hat{\mathbf{f}}^{1*}_{s_1} \circ \mathrm{d}A = \mathrm{d}A$, where the asterisk indicates push forward.
3. $(\nabla \mathbf{w})\mathbf{m} = \mathbf{0}$.
4. $\nabla_\Sigma w_m = 0$, with $w_m = \mathbf{w} \cdot \mathbf{m}$ and $\nabla_\Sigma$ the surface gradient defined above.

Two lemmata are useful for subsequent calculations.

**Lemma 1.** *For any isocoric vector field $\tilde{\mathbf{w}}(\cdot)$ of class $C^1(\mathcal{B}_0)$,*

$$\Pi \cdot \nabla_\Sigma \mathbf{w} = ((\nabla \mathbf{w})\mathbf{m}) \cdot \mathbf{m}, \tag{52}$$

*with $\mathbf{w} = \tilde{\mathbf{w}}(\mathbf{X})$.*

The result follows from direct calculation.

A second order tensor field $\Sigma \ni \mathbf{X} \longmapsto^{\tilde{\mathbf{A}}} \mathbf{A} = \tilde{\mathbf{A}}(\mathbf{X}) \in \mathrm{Hom}(\mathbb{R}^3, \mathbb{R}^3)$ is called *superficial* if $\mathbf{Am} = 0$ at each $\mathbf{X}$.

**Lemma 2.** *For any second-order superficial tensor field $\tilde{\mathbf{A}}$ on $\Sigma$, one gets*

$$\mathbf{m} \cdot \mathrm{Div}_\Sigma \mathbf{A} = \mathbf{A} \cdot \mathsf{L}, \tag{53}$$

*with $\mathsf{L}$ the curvature tensor of $\Sigma$, as defined above.*

**Definition 2.** A surface energy density $\phi$ is invariant with respect to the action of $\hat{\mathbf{f}}^1_{s_1}$ above and $\mathbf{f}^2_{s_2}$, $G$ in Section 2.3 if

$$\tilde{\phi}(\mathbf{m}, \mathbb{F}, \boldsymbol{\nu}, \mathbb{N}) = \tilde{\phi}(\nabla \hat{\mathbf{f}}^{1\mathrm{T}} \mathbf{m}, (\mathrm{grad}_\Sigma \, \mathbf{f}^2)\mathbb{F}(\nabla \hat{\mathbf{f}}^1)^{-1}, \boldsymbol{\nu}_g, \mathbb{N}_g(\nabla \hat{\mathbf{f}}^1)^{-1}), \tag{54}$$

for any $g \in G$ and $s_1, s_2 \in \mathbb{R}^+$, where $\mathbb{N}_g = \langle \nabla \boldsymbol{\nu}_g \rangle \Pi$ and we have used notations common to Definition 1.

Let $\mathfrak{X}$ be a sufficiently smooth vector density defined over $\Sigma$ by

$$\mathfrak{X} = -\phi\Pi\mathbf{w} + (\partial_{\mathbb{F}}\phi)^{\mathrm{T}}(\mathbf{v} - \langle\mathbf{F}\rangle\mathbf{w}) + (\partial_{\mathbb{N}}\phi)^{*}(\xi_{\mathcal{M}}(\boldsymbol{\nu}) - \langle\nabla\boldsymbol{\nu}\rangle\mathbf{w}) - (\partial_{\mathbf{m}}\phi \otimes \mathbf{m})\mathbf{w}, \quad (55)$$

where $(\partial_{\mathbb{F}}\phi)^{\mathrm{T}} \in \mathrm{Hom}(T_{\mathbf{x}}\mathcal{B}, T_{\mathbf{X}}\Sigma)$ and $(\partial_{\mathbb{N}}\phi)^{*} \in \mathrm{Hom}(T_{\boldsymbol{\nu}}\mathcal{M}, T_{\mathbf{X}}\Sigma)$.

The definition of $\mathfrak{X}$ (not used elsewhere) deserve physical clarifications. First we come back to (39) reminding also the definitions (30) and (31) of $\mathcal{Q}$ and $\mathfrak{F}$. In fact, Eq. (39) is a balance between the rate of a sort of 'production' of 'relative' inertia in $\mathfrak{b}_\Sigma$ and an energetic flux across its boundary $\partial\mathfrak{b}_\Sigma$. The term $(\mathbf{v} - \mathbf{F}\mathbf{w})$ is actually the difference in the current placement of the body between the virtual velocity induced by $\mathbf{f}^2$ and the push forward (by means of the gradient of deformation $\mathbf{F}$) of the virtual rate of relabeling $\mathbf{w}$. Similar interpretation has the difference $(\xi_{\mathcal{M}}(\boldsymbol{\nu}) - (\nabla\boldsymbol{\nu})\mathbf{w})$ over the manifold $\mathcal{M}$ of substructural shapes. Moreover, when multiplied by the normal $\mathbf{n}$ to $\partial\mathfrak{b}_\Sigma$, the terms $((\partial_{\mathbf{F}}\mathcal{L})^{\mathrm{T}}(\mathbf{v} - \mathbf{F}\mathbf{w})) \cdot \mathbf{n}$ and $((\partial_{\nabla\boldsymbol{\nu}}\mathcal{L})^{*}(\xi_{\mathcal{M}}(\boldsymbol{\nu}) - (\nabla\boldsymbol{\nu})\mathbf{w})) \cdot \mathbf{n}$ are respectively the power of the Piola–Kirchhoff stress and of the microstress (see also corollaries after Theorem 1) developed in the relative velocities $(\mathbf{v} - \mathbf{F}\mathbf{w})$ and $(\xi_{\mathcal{M}}(\boldsymbol{\nu}) - (\nabla\boldsymbol{\nu})\mathbf{w})$. Finally, $\mathcal{L}\mathbf{w} \cdot \mathbf{n}$ is the flux of energy across $\partial\mathfrak{b}_\Sigma$ associated with the rate of relabeling.

When the surface $\Sigma$ has energy content (as in the structured case treated here), the balance (39) is insufficient to represent the energetic landscape around $\Sigma$, so that one is pushed to insert the energetic contribution associated with the energy of $\Sigma$. Then, one needs *surface counterparts* of $\mathcal{Q}$ and $\mathfrak{F}$. Of course, the counterpart of $\mathcal{Q}$ does not exist because $\Sigma$ is not endowed with *independent* inertia. On the contrary, the counterpart of $\mathfrak{F}$ does exist because $\Sigma$ is endowed with own energy $\phi$. It coincides with $\mathfrak{X}$.

By indicating with $\tilde{\mathsf{n}}(\cdot)$ the normal vector field along the piecewise smooth curve $\partial(\mathfrak{b}_\Sigma \cap \Sigma)$ such that, at each $\mathbf{X} \in \partial(\mathfrak{b}_\Sigma \cap \Sigma)$ where it is well defined, the vector $\mathsf{n} = \tilde{\mathsf{n}}(\mathbf{X})$ belongs to the tangent plane to $\Sigma$ at $\mathbf{X}$, the product $\mathfrak{X} \cdot \mathsf{n}$ is a surface energetic flux along the tangent plane to $\Sigma$ at each $\mathbf{X} \in \partial(\mathfrak{b}_\Sigma \cap \Sigma)$ where $\mathsf{n}$ is defined. In particular, $-\phi\Pi\mathbf{w} \cdot \mathsf{n}$ is the flux of surface energy with respect to the virtual flow generated by $\mathbf{w}$, while $((\partial_{\mathbb{F}}\phi)^{\mathrm{T}}(\mathbf{v} - \langle\mathbf{F}\rangle\mathbf{w})) \cdot \mathsf{n}$ and $((\partial_{\mathbb{N}}\phi)^{*}(\xi_{\mathcal{M}}(\boldsymbol{\nu}) - \langle\nabla\boldsymbol{\nu}\rangle\mathbf{w})) \cdot \mathsf{n}$ are respectively the power of the standard surface stress and the surface microstress (as it will be clear in Theorem 2) developed in the relative velocities $(\mathbf{v} - \langle\mathbf{F}\rangle\mathbf{w})$ and $(\xi_{\mathcal{M}}(\boldsymbol{\nu}) - \langle\nabla\boldsymbol{\nu}\rangle\mathbf{w})$. Finally, the term $((\partial_{\mathbf{m}}\phi \otimes \mathbf{m})\mathbf{w}) \cdot \mathsf{n}$ is the surface power of the shear stress arising from the anisotropy of $\Sigma$.

In summary, the construction of $\mathfrak{X}$ accrues from a strict physical analogy with $\mathfrak{F}$ defined in (31).

Below, $\mathrm{d}\mathcal{H}^1$ will denote the one-dimensional Hausdorff measure along $\partial(\mathfrak{b}_\Sigma \cap \Sigma)$.

**Theorem 2.** *Let $\Sigma$ be a structured surface with surface energy $\phi$. Let us assume*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathfrak{b}_\Sigma} \mathcal{Q}\,\mathrm{d}^3\mathbf{X} + \int_{\partial\mathfrak{b}_\Sigma} \mathfrak{F} \cdot \mathbf{n}\,\mathrm{d}\mathcal{H}^2 + \int_{\partial(\mathfrak{b}_\Sigma\cap\Sigma)} \mathfrak{X} \cdot \mathsf{n}\,\mathrm{d}\mathcal{H}^1 = 0 \quad (56)$$

*for any part $\mathfrak{b}_\Sigma$ of $\mathcal{B}_0$ crossing $\Sigma$. If $\mathcal{L}$ and $\phi$ are invariant with respect to $\hat{\mathbf{f}}^1_{s_1}$, $\mathbf{f}^2_{s_2}$ and $G$, covariant pointwise balances across $\Sigma$ follow as in the list below:*

1. *The action of $\mathbf{f}^2_{s_2}$ alone implies the interfacial balance of standard interactions*

$$[\mathbf{P}]\mathbf{m} + \mathrm{Div}_\Sigma\mathbb{T} = -\rho_0[\dot{\mathbf{x}}]U, \quad (57)$$

*where* $\mathbb{T} = -\partial_{\mathbb{F}}\phi \in \mathrm{Hom}(T^*_{\mathbf{X}}\Sigma, T^*_{\mathbf{x}}\mathcal{B})$ *is the surface Piola–Kirchhoff stress.*

2. *The action of G alone implies the interfacial balance of substructural interactions*

$$[\mathcal{S}]\mathbf{m} + \mathrm{Div}_\Sigma \mathbb{S} - \mathfrak{z} = -\rho_0[\partial_{\dot{\boldsymbol{\nu}}}\chi]U, \tag{58}$$

*where* $\mathbb{S} = -\partial_{\mathbb{N}}\phi \in \mathrm{Hom}(T^*_{\mathbf{X}}\Sigma, T^*_{\boldsymbol{\nu}}\mathcal{M})$ *is the surface microstress and* $\mathfrak{z} = \partial_{\boldsymbol{\nu}}\phi \in T^*_{\boldsymbol{\nu}}\mathcal{M}$ *the surface self-force.*

3. *The action of* $\hat{\mathbf{f}}^1_{s_1}$ *alone implies the interfacial configurational balance along the normal* $\mathbf{m}$ *in absence of dissipative forces driving* $\Sigma$*, namely*

$$\begin{aligned}&\mathbf{m}\cdot[\mathbb{P}]\mathbf{m} + \mathbb{C}_{\tan}\cdot \mathsf{L} + \mathrm{Div}_\Sigma \mathfrak{c}\\ &\quad = \rho_0 U[(\nabla\boldsymbol{\nu})^*\partial_{\dot{\boldsymbol{\nu}}}\chi]\cdot\mathbf{m} + \rho_0[\chi(\boldsymbol{\nu},\dot{\boldsymbol{\nu}})] - \tfrac{1}{2}\rho_0 U^2[|\mathbf{F}\mathbf{m}|^2],\end{aligned} \tag{59}$$

*where*

$$\mathbb{C}_{\tan} = \phi\Pi - \mathbb{F}^{\mathrm{T}}\mathbb{T} - \mathbb{N}^*\mathbb{S} \tag{60}$$

*is a generalized version of the surface Eshelby stress and*

$$\mathfrak{c} = -\partial_{\mathbf{m}}\phi - \mathbb{T}^{\mathrm{T}}\langle\mathbf{F}\rangle\mathbf{m} - \mathbb{S}^*\langle\nabla\boldsymbol{\nu}\rangle\mathbf{m} \tag{61}$$

*is a surface shear.*

By following different procedures, Eq. (57) has been derived in [17] while Eqs. (58) and (59) in [22,23] (the version of (59) for simple bodies is clearly obtained in [16]). Here, with Theorem 2 we prove their covariance: this is the main novelty of the theorem itself.

### 4.3. Proof of Theorem 2

**Step 1.** In accord to Definition 2, since $\tilde{\phi}$ is invariant, we have

$$\frac{\mathrm{d}}{\mathrm{d}s_1}\tilde{\phi}(\nabla\hat{\mathbf{f}}^{1\mathrm{T}}\mathbf{m}, (\mathrm{grad}_\Sigma\,\mathbf{f}^2)\mathbb{F}(\nabla\hat{\mathbf{f}}^1)^{-1}, \boldsymbol{\nu}_g, \mathbb{N}_g(\nabla\hat{\mathbf{f}}^1)^{-1})|_{s_1=0,s_2=0,s_3=0} = 0, \tag{62}$$

$$\frac{\mathrm{d}}{\mathrm{d}s_2}\tilde{\phi}(\nabla\hat{\mathbf{f}}^{1\mathrm{T}}\mathbf{m}, (\mathrm{grad}_\Sigma\,\mathbf{f}^2)\mathbb{F}(\nabla\hat{\mathbf{f}}^1)^{-1}, \boldsymbol{\nu}_g, \mathbb{N}_g(\nabla\hat{\mathbf{f}}^1)^{-1})|_{s_1=0,s_2=0,s_3=0} = 0, \tag{63}$$

$$\frac{\mathrm{d}}{\mathrm{d}s_3}\tilde{\phi}(\nabla\hat{\mathbf{f}}^{1\mathrm{T}}\mathbf{m}, (\mathrm{grad}_\Sigma\,\mathbf{f}^2)\mathbb{F}(\nabla\hat{\mathbf{f}}^1)^{-1}, \boldsymbol{\nu}_g, \mathbb{N}_g(\nabla\hat{\mathbf{f}}^1)^{-1})|_{s_1=0,s_2=0,s_3=0} = 0, \tag{64}$$

which correspond respectively to

$$\mathbb{F}^{\mathrm{T}}\mathbb{T}\cdot\nabla_\Sigma\mathbf{w} + \mathbb{N}^*\mathbb{S}\cdot\nabla_\Sigma\mathbf{w} + \partial_{\mathbf{m}}\phi\cdot(\nabla\mathbf{w})\mathbf{m} = 0, \tag{65}$$

$$\mathbb{T}\cdot\nabla_\Sigma\mathbf{v} = 0, \tag{66}$$

$$\mathfrak{z}\cdot\xi_{\mathcal{M}}(\boldsymbol{\nu}) + \mathbb{S}\cdot\nabla_\Sigma\xi_{\mathcal{M}}(\boldsymbol{\nu}) = 0. \tag{67}$$

They will be useful tools below.

**Step 2.** If we shrink $\mathfrak{b}_\Sigma$ to $\mathfrak{b}_\Sigma \cap \Sigma$ uniformly in time, transport and Gauss theorems (see also (40) and (41)) allow us to obtain the pointwise balance

$$-[\mathcal{Q}]U + [\mathfrak{F}] \cdot \mathbf{m} + \mathrm{Div}_\Sigma \mathfrak{X} = 0, \tag{68}$$

thanks to the arbitrariness of $\mathfrak{b}_\Sigma$. Of course, the sole difference between (68) and (42) is the term $\mathrm{Div}_\Sigma \mathfrak{X}$ accounting for the interfacial structure of the surface $\Sigma$.

**Step 3.** Deduction of the referential interfacial balance of standard interactions (57). If $\mathbf{f}^2$ acts alone, then

$$\mathfrak{X} = \mathbb{T}^{\mathrm{T}}\mathbf{v}, \quad \mathcal{Q} = \rho\dot{\mathbf{x}} \cdot v, \quad \mathfrak{F} = -\mathbf{P}^{\mathrm{T}}\mathbf{v}. \tag{69}$$

Moreover, thanks to (66) we get

$$\mathrm{Div}_\Sigma \mathfrak{X} = \mathbf{v} \cdot \mathrm{Div}_\Sigma \mathbb{T}. \tag{70}$$

Then, from (68) we obtain (57) thanks to the arbitrariness of $\mathbf{v}$, which is continuous across $\Sigma$.

**Step 4.** Deduction of the referential interfacial balance of substructural interactions (58).

If $G$ acts alone, then

$$\mathfrak{X} = \mathbb{S}^* \xi_{\mathcal{M}}(\boldsymbol{\nu}), \quad \mathcal{Q} = \rho_0 \partial_{\dot{\boldsymbol{\nu}}} \chi \cdot \boldsymbol{\xi}_{\mathcal{M}}(\boldsymbol{\nu}), \quad \mathfrak{F} = -\mathcal{S}^* \xi_{\mathcal{M}}(\boldsymbol{\nu}). \tag{71}$$

Moreover, thanks to (67) we get

$$\mathrm{Div}_\Sigma \mathfrak{X} = \xi_{\mathcal{M}}(\boldsymbol{\nu}) \cdot (\mathrm{Div}_\Sigma \mathbb{S} - \mathfrak{z}). \tag{72}$$

Then, from (68) we obtain (58) thanks to the arbitrariness of the element $\xi$ selected in the Lie algebra of $G$.

**Step 5.** Deduction of the balance of configurational forces along the normal $\mathbf{m}$ (i.e. (59)). If $\hat{\mathbf{f}}^1$ acts alone, then

$$\mathcal{Q} = -\rho\mathbf{F}^{\mathrm{T}}\dot{\mathbf{x}} \cdot \mathbf{w} - \rho_0 (\nabla \boldsymbol{\nu})^* \partial_{\dot{\boldsymbol{\nu}}} \chi \cdot \mathbf{w}, \tag{73}$$

$$\mathfrak{F} = ((\tfrac{1}{2}\rho_0 |\dot{\mathbf{x}}|^2 + \rho_0 \chi(\boldsymbol{\nu}, \dot{\boldsymbol{\nu}}))\mathbf{I} - \mathbb{P})\mathbf{w} \tag{74}$$

and, after some algebra,

$$\mathfrak{X} = -\mathbb{C}_{\tan}^{\mathrm{T}}\mathbf{w} - \mathfrak{c} w_m \tag{75}$$

with $\mathbb{C}_{\tan}$ and $\mathfrak{c}$ defined respectively by (60) and (61) and $w_m = \mathbf{w} \cdot \mathbf{m}$.

Now, Eq. (68) comes into play: we will evaluate the component along $\mathbf{m}$ of the right-hand side term of (68), a vector, by taking also into account the arbitrariness of $\mathbf{w}$.

First we write

$$\begin{aligned} -[\mathcal{Q}]U + [\mathfrak{F}] \cdot \mathbf{m} = {} & \rho_0 [\mathbf{F}^{\mathrm{T}}\dot{\mathbf{x}}] U \cdot \mathbf{w} + \rho_0 [(\nabla \boldsymbol{\nu})^* \partial_{\dot{\boldsymbol{\nu}}} \chi] U \cdot \mathbf{w} \\ & + \tfrac{1}{2}\rho_0 [|\dot{\mathbf{x}}|^2] \mathbf{w} \cdot \mathbf{m} + \rho_0 [\chi(\boldsymbol{\nu}, \dot{\boldsymbol{\nu}})] \mathbf{w} \cdot \mathbf{m} - [\mathbb{P}] \mathbf{w} \cdot \mathbf{m}. \end{aligned} \tag{76}$$

Moreover, by using (65) and Lemma 1, we also get

$$\mathrm{Div}_\Sigma (\mathbb{C}_{\tan}^{\mathrm{T}}\mathbf{w} + \mathfrak{c} w_m) = \mathbf{w} \cdot (\mathrm{Div}_\Sigma \mathbb{C}_{\tan} + (\mathrm{Div}_\Sigma \mathfrak{c})\mathbf{m}). \tag{77}$$

Of course, in obtaining (77), properties 3 and 4 of the definition of the relabeling $\hat{\mathbf{f}}^1$ of $\mathcal{B}_0$ including $\Sigma$ play a crucial role.

By inserting (76) and (77) in (68), thanks to the arbitrariness of $\mathbf{w}$, we obtain

$$\rho_0[\mathbf{F}^{\mathrm{T}}\dot{\mathbf{x}}]U + \rho_0[(\nabla \boldsymbol{\nu})^* \partial_{\dot{\boldsymbol{\nu}}} \chi]U + \tfrac{1}{2}\rho_0[|\dot{\mathbf{x}}|^2]\mathbf{m} + \rho_0[\chi(\boldsymbol{\nu}, \dot{\boldsymbol{\nu}})]\mathbf{m}$$
$$= [\mathbb{P}^{\mathrm{T}}]\mathbf{m} + \mathrm{Div}_\Sigma \mathbb{C}_{\tan} + (\mathrm{Div}_\Sigma \mathbf{c})\mathbf{m} \tag{78}$$

and we shall evaluate the component along $\mathbf{m}$ of (78).

First we focus our attention on terms involving $\dot{\mathbf{x}}$ and $\dot{\boldsymbol{\nu}}$. Let us introduce the averaged velocity $\overline{\mathbf{v}}$ given by

$$\overline{\mathbf{v}} = \langle \dot{\mathbf{x}} \rangle + U\langle \mathbf{F} \rangle \mathbf{m}. \tag{79}$$

With the help of the relation $[\dot{\mathbf{x}}] = -U[\mathbf{F}]\mathbf{m}$ introduced previously, we then get

$$\rho_0[\mathbf{F}^{\mathrm{T}}\dot{\mathbf{x}}]U \cdot \mathbf{m} = \rho_0[\dot{\mathbf{x}}] \cdot \overline{\mathbf{v}} - \rho_0[|\dot{\mathbf{x}}|^2]; \tag{80}$$

in other words the normal component of the vector $\rho_0[\mathbf{F}^{\mathrm{T}}\dot{\mathbf{x}}]U$ is equal to *minus* the jump of the relative kinetic energy $(1/2)[\rho_0|\dot{\mathbf{x}} - \overline{\mathbf{v}}|^2]$ as it is simple to verify.

Still taking into account the relation $[\dot{\mathbf{x}}] = -U[\mathbf{F}]\mathbf{m}$ and the definition of $\overline{\mathbf{v}}$, we also find

$$\tfrac{1}{2}\rho_0[|\dot{\mathbf{x}}|^2] = -\rho_0[\dot{\mathbf{x}}] \cdot \overline{\mathbf{v}} + \tfrac{1}{2}U^2[|\mathbf{F}\mathbf{m}|^2]. \tag{81}$$

Now, by evaluating the normal component of (78), using (80), (81) and taking into account that $\mathbf{m} \cdot \mathrm{Div}_\Sigma \mathbb{C}_{\tan} = \mathbb{C}_{\tan} \cdot \mathsf{L}$ as a consequence of Lemma 2, we obtain (59) and the theorem is proven.

**Remark 11.** (Fields of applicability of Proposition 1 and Theorem 2). The results collected in Proposition 1 and Theorem 2 can be applied in various physical circumstances in condensed matter physics, involving the evolution of sharp defects. In particular, when $\Sigma$ evolves irreversibly a dissipative driving force must be added to (45) and (59) (the physical reasons are clearly explained in [16,17] for simple bodies; the case of complex bodies is treated in [22,23]). Examples are listed below.

1. Cracks in complex bodies.
2. Sharp interfaces between paraelectric and ferroelectric phases.
3. Evolution of sharp damage fronts.
4. Sharp interfaces between isotropic and oriented (e.g. nematic) phases (e.g. in liquid crystals).
5. Solidification of complex fluids.
6. Growing defects in biological tissues.

## References

[1] R. Abraham, J.E. Marsden, Foundations of Mechanics, The Benjamin/Cummings Publishing Company, London, 1978.
[2] H. Brezis, Y. Li, Topology and Sobolev spaces, J. Funct. Anal. 183 (2001) 321–369.
[3] G. Capriz, Continua with latent microstructure, Arch. Ration. Mech. Anal. 90 (1985) 43–56.
[4] G. Capriz, Continua with Microstructure, Springer-Verlag, Berlin, 1989.

[5] G. Capriz, Continua with substructure. Part I and Part II, Phys. Mesomech. 3 (2001) 5–14 and 37–50.
[6] G. Capriz, Elementary preamble to a theory of granular gases, Rend. Sem. Mat. Univ. Padova 110 (2003) 179–198.
[7] G. Capriz, P. Giovine, On microstructural inertia, Math. Mod. Meth. Appl. Sci. 7 (1997) 211–216.
[8] G. Capriz, P.M. Mariano, Balance at a junction in coherent interfaces in continua with substructure, in: G. Capriz, P.M. Mariano (Eds.), Advances in Multifield Theories for Continua with Substructure, Birkhauser, Boston, 2003, pp. 243–263.
[9] G. Capriz, P.M. Mariano, Symmetries and Hamiltonian formalism for complex materials, J. Elasticity 72 (2003) 57–70.
[10] E. Cosserat, F. Cosserat, Sur la théorie des corps deformables, Dunod, Paris, 1909.
[11] P.G. de Gennes, J. Prost, The Physics of Liquid Crystals, Clarendon Press, 1993.
[12] G. Del Piero, D.R. Owen, Structured deformation of continua, Arch. Ration. Mech. Anal. 124 (1993) 99–155.
[13] G. Del Piero, D.R. Owen,Structured Deformation, Quaderni dell'Istituto di Alta Matematica, Florence, 2000.
[14] J.L. Ericksen, Liquid crystals with variable degree of orientation, Arch. Ration. Mech. Anal. 113 (1991) 97–120.
[15] R.C. Fetecau, J.E. Marsden, M. West, Variational multisymplectic formulation of non-smooth continuum mechanics, in: E. Kaplan, J.E. Marsden, K.R. Sreenivasan (Eds.), Perspective and Problems in Nonlinear Sciences, Springer, Berlin, 2003, pp. 229–261.
[16] M.E. Gurtin, The nature of configurational forces, Arch. Ration. Mech. Anal. 131 (1995) 67–100.
[17] M.E. Gurtin, A. Struthers, Multiphase thermomechanics with interfacial structure. III. Evolving phase boundaries in the presence of bulk deformation, Arch. Ration. Mech. Anal. 112 (1990) 97–160.
[18] F. Hang, F. Lin, Topology of Sobolev mappings III, Comm. Pure Appl. Math. 61 (2003) 1383–1415.
[19] S. Helgason, Differential Geometry Lie Groups and Symmetric Spaces, Academic Press, New York, 1978.
[20] F.M. Leslie, Some constitutive equations for liquid crystals, Arch. Ration. Mech. Anal. 28 (1968) 265–283.
[21] P.M. Mariano, Some remarks on the variational description of microcracked bodies, Int. J. Non-Lin. Mech. 34 (1999) 633–642.
[22] P.M. Mariano, Configurational forces in continua with microstructure, Z. Angew. Math. Phys. ZAMP 51 (2000) 752–791.
[23] P.M. Mariano, Multifield theories in mechanics of solids, Adv. Appl. Mech. 38 (2001) 1–93.
[24] P.M. Mariano, Some thermodynamical aspects of the solidification of two-phase flows, Meccanica 39 (2004) 369–382.
[25] P.M. Mariano, Elements of Multifield Theories of Complex Bodies, Birkhauser, Boston, in preparation.
[26] P.M. Mariano, F.L. Stazi, Strain localization due to crack-microcrack interaction: X-FEM for a multifield approach, Comp. Meth. Appl. Mech. Eng. 193 (2004) 5035–5062.
[27] J.E. Marsden, T.R.J. Hughes, Mathematical Foundations of Elasticity, Prentice-Hall, New Jersey, 1983.
[28] R.D. Mindlin, Micro-structures in linear elasticity, Arch. Ration. Mech. Anal. 16 (1964) 51–78.
[29] W. Noll, A mathematical theory of the mechanical behavior of continuous media, Arch. Ration. Mech. Anal. 2 (1958) 197–226.
[30] W. Noll, A new mathematical theory of simple materials, Arch. Ration. Mech. Anal. 48 (1972) 1–50.
[31] S.B. Rochal, V.L. Lorman, Minimal model of the phonon–phason dynamics in icosahedral quasicrystls and its application to the problem of internal friction in the i-AlPdMn alloy, Phys. Rev. B 66 (144204) (2001) 1–9.
[32] R. Segev, A geometrical framework for the statics of materials with microstructure, Mater. Models Meth. Appl. Sci. 4 (1994) 871–897.
[33] R.W. Sharpe, Differential Geometry, Springer-Verlag, Berlin, 1997.
[34] M. Silhavy, A condition equivalent to the existence of non-equilibrium entropy and temperature for materials with internal variables, Arch. Ration. Mech. Anal. 68 (1978) 299–332.
[35] M. Silhavy, The Mechanics and Thermodynamics of Continuous Media, Springer, Berlin, 1997.